Efficient Discrete-Time Dynamics of Geometrically Exact Beams Based on Relative Kinematics

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EXTENDED ABSTRACT

1 Introduction

Geometrically exact beam theories have become widespread in literature to accurately model flexible structures undergoing large deformations [1]. A powerful property of these models is that their configuration space is a Lie group; more precisely, in the case of a general three-dimensional Simo-Reissner beam with all six deformation modes, it is an (infinite-dimensional) product of the Special Euclidean group $\mathsf{SE}(3)$. Among the variety of literature in this field, we take the viewpoint of discrete geometric mechanics and variational integrators in this work to obtain fully discrete dynamical models of such beams. Variational integrators are momentum-preserving, symplectic and can preserve the Lie group structure. For beams, these have been derived previously in the form of Lie group variational integrators (LGVIs), both from the traditional Lagrangian viewpoint (involving subsequent discretization first in space, then in time) that is also considered here [2] and from the field theoretic, covariant viewpoint [3], that involves simultaneous space-time discretization.

These approaches are all based on an *absolute* kinematic formulation, meaning that the configuration of the beam is expressed w.r.t. a fixed inertial frame. In contrast, we introduce a variational integrator based on a *relative* kinematic description, which exploits the serial chain structure of the spatially discretized beam. Here, we use the relative displacements of the discrete cross-sections w.r.t. their respective predecessors in the chain as configuration variables – similarly to how the relative joint angles are used as configuration variables in rigid robotics. This choice is motivated by the following main points:

1. The resulting models (both in continuous and discrete time) have a minimal number of states (as opposed to *global* approaches that use elements of $\mathsf{SE}(3)$ or non-minimal parameterizations such as quaternions as configuration variables), while not suffering from relevant drawbacks such as singularities.

2. Due to the generality of the beam models, numerical efficiency is still an ongoing focus of research. This especially concerns the case where the general model is used to simulate e.g., slender, stiff beams, in which the effects of shear and elongation become negligible. However, since these effects are still present within the model, they introduce stiff, high-frequency modes into the system, leading to the requirement of small time steps and/or fine spatial discretization. In the present approach, these modes can be elegantly excluded by restricting the free displacement variables to a set of allowed deformation modes [4]. Other existing solutions include using constraints (leading to DAE systems) or specific parameterizations for the beam configuration [5].

3. The structure of the resulting model is analogous to models in classical rigid robotics, thus simplifying the consistent modeling of multibody systems involving both rigid and elastic elements. This concerns both the kinematic description (including using a Jacobian matrix for velocity kinematics) and the structure of the equations of motion, making this approach especially appealing for robotics and control.

In this work, we derive and analyze a variational integrator for geometrically exact beams in the relative kinematic description. In continuous time, using relative deformation variables as states has been originally proposed by [6] in the more practical context of soft robotics. Building on this idea, we perform the derivation from a purely variational point of view in the framework of geometric mechanics, under consistent consideration of the Lie group structure. Finally, we make the transition to discrete time, leading to an elegant and straight-forward formulation of the relative kinematics and the discrete-time beam dynamics.

2 Relative Beam Dynamics in Continuous and Discrete Time

In the standard approach [2, 3], a geometrically exact beam is modeled as a Cosserat continuum in three-dimensional space. The beam's configuration $g : [0, L] \to \mathsf{SE}(3)$, parameterized by the arc length along the undeformed reference configuration, is described by the positions and attitudes of its infinitesimally thin cross sections that are stacked along the beam's center line. The body-fixed velocity and deformation gradient along the beam are given by the left-trivialized, convective derivatives $\hat{\eta} = g^{-1}\dot{g}$ and $\hat{\xi} = g^{-1}g'$ in the Lie algebra $\mathfrak{se}(3)$. The beam's continuous Lagrangian can be written the standard form

$$L(g, \eta, \xi) = \int_0^L \frac{1}{2}\eta^T \mathbb{M}\eta - \frac{1}{2}(\xi - \bar{\xi})^T \mathbb{C}(\xi - \bar{\xi}) - \mathcal{U}(g)\, \mathrm{d}s \tag{1}$$

with quadratic kinetic and deformation energy terms (defined with appropriate mass and stiffness tensors), reference deformations $\bar{\xi}$ and the potential energy term due to gravity $\mathcal{U}(g)$. With suitably defined variations of g, $\hat{\eta}$ and $\hat{\xi}$ on the Lie group, one can then derive the fully continuous PDE beam model in Lagrangian form. For the spatial discretization, the overall beam is divided into n discrete segments, which are bounded by *nodes* at the two ends. Now, a *spatially discrete Lagrangian* is constructed by approximating the beam's energy over one segment of the discretized beam using a quadrature rule; the overall Lagrangian is then approximated as the sum of all segment approximations. Using the Lie group exponential map $\exp : \mathfrak{g} \to G$, the deformation gradient $\hat{\xi}$ over a segment $(a, a+1)$ is discretized as the Lie algebra element $\hat{\xi}_a = \exp^{-1}(g_a^{-1} g_{a+1})$ generating the relative deformation of node $a+1$ w.r.t. node a. Since $\hat{\xi}_a$ is constant over a segment, this naturally leads to the assumption of *piecewise constant strains* in [6].

At this point, we transition to the description in relative variables. First, we recursively express the configuration of a node $a+1$ in terms of the configuration of the previous node a and the relative deformation with $g_{a+1} = g_a \exp(\hat{\xi}_a)$. By substitution into the definition of the velocity $\hat{\eta}_a$ of a node a, we can express the velocity in terms of discrete deformation variables using the right-trivialized derivative of the exponential map [3]. Using the Lie algebra isomorphism $\mathfrak{se}(3) \simeq \mathbb{R}^6$, this relationship can take the familiar form $\eta_a = J_a(\boldsymbol{\xi})\dot{\boldsymbol{\xi}}$, where $\boldsymbol{\xi} := [\xi_0, ..., \xi_{n-1}] \in \mathbb{R}^{6n}$ and $J_a(\boldsymbol{\xi}) \in \mathbb{R}^{6\times 6n}$ is the *geometric Jacobian* of node a. To derive continuous-time equations of motion, we next compute the variations of g_a and $\hat{\eta}_a$ expressed as variations in ξ_a by using the previously defined kinematic relationships. Applying Hamilton's principle, substituting the relative variations and velocity expressions, performing a number of manipulations and finally integrating by parts in time, we then arrive at the continuous-time equations of motion. These have the usual second-order form of rigid robots with a configuration-dependent, nonlinear mass matrix and nonlinear Coriolis and gravity terms and an additional, linear term for the elastic deformation.

We now transfer the relative formulation to discrete time via the discrete geometric mechanics principles. As in the standard approach for variational integrators, we define a fully discrete Lagrangian approximating the action of the temporally continuous Lagrangian over one time step using a quadrature rule (in analogy to the previous spatial discretization). To this end, we define a discrete velocity over a time step $(k, k+1)$ with sample time h as the simple difference quotient $\eta_a^k = J_a(\boldsymbol{\xi}^k)\left(\frac{\xi^{k+1} - \xi^k}{h}\right)$. Computing the discrete-time variations, we can then apply the discrete Hamilton principle and obtain a variational integrator for the geometrically exact beam in relative description. As typical for variational integrators, the resulting equations are implicit and need to be solved numerically in each time step. A remarkable feature of the resulting integrator is that it is defined purely in Euclidean variables $\boldsymbol{\xi}$, which is in contrast to the more complex LGVIs obtained for beam models in absolute description.

Finally, we reduce the free deformation variables to exclude unwanted deformation modes, resulting in a lower-order model. As proposed in [4], we introduce the decomposition $\xi_a = B\psi_a + \tilde{B}\tilde{\psi}_a$, where B and $\tilde{B}$ are complementary selection matrices of appropriate dimension, and ψ_a and $\tilde{\psi}_a$ are the *allowed* and *constrained* (i.e., constant) deformation vectors. Only the allowed deformations ψ_a are retained as variables in the dynamical model, while the constrained deformations remain fixed over time. This expression can be incorporated into the derivation of the continuous and discrete-time models in a straight-forward way, which allows to elegantly derive a lower-order model for geometrically exact beams with a reduced number of deformation modes, thus avoiding the aforementioned numerical problems.

3 Numerical Evaluation

The proposed relative model in continuous and discrete time is evaluated numerically with respect to its accuracy and numerical efficiency. Of special interest is the efficiency in comparison to the previously mentioned models in absolute description; in particular for the critical cases of high numerical stiffness due to negligible deformation modes.

4 Summary and Conclusion

In this contribution, we introduced a discrete-time model for the dynamics of geometrically exact beams based on a relative kinematic description. This is advantageous from a computational point of view, as it results in a *minimal* model; furthermore, negligible deformation modes can be elegantly excluded to avoid the numerical issues associated to them. Finally, the structure of the resulting model makes it particularly well-suited for applications in robotics and control.

References

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