On optimal control problems in redundant coordinates

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EXTENDED ABSTRACT

1 Introduction

The present work focuses on optimal control problems of mechanical systems subject to holonomic constraints. Here, the motion of the system is governed by differential algebraic equations (DAEs) which are typically of index three but can be reduced to index two easily, see e.g. [1, 2]. For some problems minimal coordinates can be found or the holonomic constraints can be eliminated by applying a nullspace method, see e.g. [3]. However, for more involved multibody systems it can be cumbersome or even impossible to find minimal coordinates. Therefore, the most general approach to the optimal control of multibody systems relies on using redundant coordinates. Concerning optimal control problems with DAEs as state equations, there are various approaches how to tackle the problem, see e.g. [4, 5, 6, 7]. In the present work we compare these alternative approaches. In this connection, a new variational integrator is presented and eventually the different approaches are compared with regard to their numerical results, gained by applying the proposed integrator.

2 The constraints of the optimal control problem

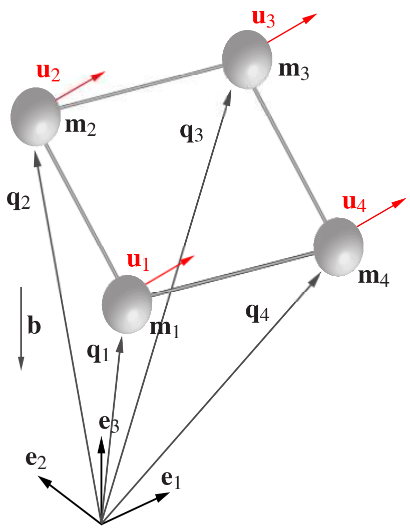

Figure 1: Four linked masses

A simple but representative example of a multibody system, who's minimal coordinates are cumbersome to find and who's nullspace matrix is cumbersome to compute is depicted in Figure 1. The four mass points m_i, $(i = 1, \dots, 4)$ are assumed to be linked by four massless rigid bars. Let

$$d\mathbf{q}_i = \mathbf{q}_{i+1} - \mathbf{q}_i \tag{1}$$

describe the distance between the masses m_{i+1} and m_i. Then four holonomic constraints on position level of the form

$$g^3(\mathbf{q}) = \frac{1}{2}(d\mathbf{q}_i^T d\mathbf{q}_i - l_0^2) = 0 \tag{2}$$

can be used to enforce the constant distance l_0 between the respective mass points. If $\mathbf{p}_i$ denotes the conjugate momentum given by $\mathbf{p}_i = m_i \dot{\mathbf{q}}_i$, the constraints on velocity level are of the form

$$g^2(\mathbf{q},\mathbf{p}) = d\mathbf{q}_i^T \left(\frac{1}{m_{i+1}} \mathbf{p}_{i+1} - \frac{1}{m_i} \mathbf{p}_i \right) \tag{3}$$

The position vectors $\mathbf{q}_i$ and the momentum vectors $\mathbf{p}_i$ are collected in associated system vectors $\mathbf{q}$ and $\mathbf{p}$, respectively.

Similarly, the constraint functions on position and velocity level are arranged in vector-valued functions $\mathbf{g}^3(\mathbf{q})$ and $\mathbf{g}^2(\mathbf{q},\mathbf{p})$, respectively. The controlled equations of motion can now be written as

$$\left.\begin{array}{rcl} \dot{\mathbf{q}} & = & \partial_{\mathbf{p}} H_j^M(\mathbf{q},\mathbf{p},\mathbf{y}) \\ \dot{\mathbf{p}} & = & -\partial_{\mathbf{q}} H_j^M(\mathbf{q},\mathbf{p},\mathbf{y}) + \mathbf{B}(\mathbf{q})\mathbf{u} \end{array}\right\} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{u}) \tag{4a}$$

$$\mathbf{0} = \partial_{\mathbf{y}} H_j^M(\mathbf{q},\mathbf{p},\mathbf{y}) \tag{4b}$$

with H_j^M being the Hamiltonians

$$H_3^M(\mathbf{q},\mathbf{p},\mathbf{y}) = T(\mathbf{q},\mathbf{p}) + V(\mathbf{q}) + \hat{\mathbf{y}}^T \mathbf{g}^3(\mathbf{q}) \tag{5}$$

$$H_2^M(\mathbf{q},\mathbf{p},\mathbf{y}) = T(\mathbf{q},\mathbf{p}) + V(\mathbf{q}) + \hat{\mathbf{y}}^T \mathbf{g}^3(\mathbf{q}) + \bar{\mathbf{y}}^T \mathbf{g}^2(\mathbf{q},\mathbf{p}) \tag{6}$$

respectively. Here, label $j \in 2,3$ indicates the differentiation index of the DAEs (4) resulting from choosing either H_3^M or H_2^M. Note that the Lagrangian multipliers are collected in vector $\mathbf{y}$. Those of them who are related to constraints $\mathbf{g}^3(\mathbf{q})$ and $\mathbf{g}^2(\mathbf{q},\mathbf{p})$ are denoted by $\hat{\mathbf{y}} \in \mathbb{R}^4$ and $\bar{\mathbf{y}} \in \mathbb{R}^4$, respectively. Moreover, T and V denote the kinetic and potential energy. In (4), $\mathbf{u}$ contains the control inputs and $\mathbf{B}$ denotes the control distribution matrix.

3 The Optimal Control Problem

Let the optimal control problem seek to minimize the cost functional

$$\mathscr{S}(\mathbf{u},\mathbf{q}) = \int_{t_0}^{t_f} C(\mathbf{u},\mathbf{q})\,\mathrm{d}t \tag{7}$$

subject to the constraints (4), which need to be satisfied during the time interval $[t_0,\ t_f]$. We further introduce the second time derivative of the constraints $\mathbf{g}^3$ and denote them by $\mathbf{g}^1(\mathbf{x},\mathbf{y},\mathbf{u})$, where the state vector $\mathbf{x}$ contains both $\mathbf{q}$ and $\mathbf{p}$. Similarly to the equations of motion (4), the necessary optimality conditions of the optimal control problem can be written in terms of the Hamiltonians

$$\mathscr{H}_{3,1}(\cdot) = \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x},\hat{\mathbf{y}},\mathbf{u}) + \boldsymbol{\eta}_1^T \mathbf{g}^1(\mathbf{x},\hat{\mathbf{y}},\mathbf{u}) - C(\mathbf{u}) \tag{8}$$

$$\mathscr{H}_{2,1}(\cdot) = \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{u}) + \boldsymbol{\eta}_1^T \mathbf{g}^1(\mathbf{x},\mathbf{y},\mathbf{u}) + \boldsymbol{\eta}_2^T \mathbf{g}^2(\mathbf{x},\mathbf{y}) - C(\mathbf{u}) \tag{9}$$

$$\mathscr{H}_{3,3}(\cdot) = \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x},\hat{\mathbf{y}},\mathbf{u}) + \boldsymbol{\eta}_3^T \mathbf{g}^3(\mathbf{x}) - C(\mathbf{u}) \tag{10}$$

$$\mathscr{H}_{2,2}(\cdot) = \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{u}) + \boldsymbol{\eta}_2^T \mathbf{g}^2(\mathbf{x}) - C(\mathbf{u}) \tag{11}$$

In particular, employing one of the above Hamiltonians yields the corresponding necessary optimality conditions

$$\dot{\mathbf{x}} = \partial_{\boldsymbol{\lambda}} \mathscr{H}_{j,k}(\mathbf{x},\mathbf{y},\mathbf{u},\boldsymbol{\lambda},\boldsymbol{\eta}) \tag{12a}$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}) \tag{12b}$$

$$\dot{\boldsymbol{\lambda}} = -\partial_{\mathbf{x}} \mathscr{H}_{j,k}(\mathbf{x},\mathbf{y},\mathbf{u},\boldsymbol{\lambda},\boldsymbol{\eta}) \tag{12c}$$

$$\mathbf{0} = \partial_{\mathbf{y}} \mathscr{H}_{j,k}(\mathbf{x},\mathbf{y},\mathbf{u},\boldsymbol{\lambda},\boldsymbol{\eta}) \tag{12d}$$

$$\mathbf{0} = \partial_{\mathbf{u}} \mathscr{H}_{j,k}(\mathbf{x},\mathbf{y},\mathbf{u},\boldsymbol{\lambda},\boldsymbol{\eta}) \tag{12e}$$

which are comprised of the state DAEs (12a), (12b), the adjoint DAEs (12c), (12d) and the optimality conditions (12e). Note that $\boldsymbol{\lambda}$ and $\boldsymbol{\eta}$ contain the adjoint variables. Moreover, depending on the choice of Hamiltonian $\mathscr{H}_{j,k}$, alternative optimality conditions are generated. The indices of $\mathscr{H}_{j,k}$ indicate that the resulting state DAEs have index j, while the adjoint DAEs have index k.

4 Outline of the talk

In the talk, we shall investigate the alternative choices of the Hamiltonians $\mathscr{H}_{j,k}$ in more detail and relate them to previous works. In addition to that, we investigate numerical methods based on the alternative Hamiltonians and compare their results in the context of representative examples dealing with mechanical systems subject to holonomic constraints.

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