Port-Hamiltonian formulation and structure-preserving discretization of hyperelastic strings

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EXTENDED ABSTRACT

1 Introduction

Port-Hamiltonian (PH) systems provide a framework for modeling, analysis and control of complex dynamical systems [1], where the complexity might result from multi-physical couplings, non-trivial domains and diverse nonlinearities. A major benefit of the PH representation is the explicit formulation of power interfaces, so-called ports, which allow for an intrinsically power-preserving interconnection of subsystems. In this way, the modular composition of a model can be facilitated. Since many technical (sub-)systems are modeled by partial differential equations (PDE), the theory of infinite-dimensional PH systems has been extended in recent years [2]. Among examples from various physical disciplines, PH formulations have also been proposed in structural mechanics, e.g. for flexible multibody dynamics [3]. With respect to the numerical discretization, the mixed finite element method has been employed to establish approximate models under preservation of the PH structure (e.g. in [4]). A core characteristic of structure-preserving discretization of PH systems is retaining the ports (e.g. pairs of velocities and forces) with their causality (which refers to the definition of boundary input variables in the system theoretic sense) on the discrete level.

In our talk, we focus on a specific one-dimensional example from nonlinear continuum mechanics. String elements [5] occur as interconnected subsystems in a wide range of applications including harbor cranes, cable cars, bionic robotic hands, underwater cables, satellite systems and much more, which motivates their PH formulation. Based on our previous work on modeling and structure-preserving discretization of geometrically nonlinear strings with linear material laws [6], we present the case of hyperelastic materials, which greatly expands the field of application.

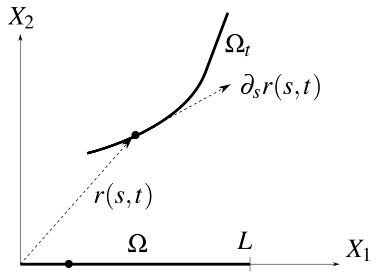

Figure 1: Material and spatial string configurations.

2 Port-Hamiltonian string representation

We consider a one-dimensional undeformed (material) string configuration $\Omega = [0, L] \subset \mathbb{R}^3$ with length $L \in \mathbb{R}$ and its current (spatial) configuration $\Omega_t \subset \mathbb{R}^3$, which is described by the position vector $r(s,t)$ depending on the material coordinate $s = X_1 \in \mathbb{R}$ and the time $t \in \mathbb{R}$, see Figure 1 for an illustration on $\mathbb{R}^2$. The balance of linear momentum of the string (see e.g. [7]) in material coordinates is given by

$$\rho A \ddot{r} = \partial_s n + b \tag{1}$$

and includes the density $\rho \in \mathbb{R}$, the cross-sectional area $A \in \mathbb{R}$, the volume forces $b \in \mathbb{R}^3$, and the contact force $n \in \mathbb{R}^3$. We consider hyperelastic materials with a stored energy density $W : \mathbb{R} \to \mathbb{R}$ depending on the strain type quantity $C = \partial_s r \cdot \partial_s r$. Thus, the contact force n and the corresponding tension $N \in \mathbb{R}$ are obtained via the constitutive relation

$$n = N \frac{\partial_s r}{||\partial_s r||} = 2\nabla W(C) \partial_s r = S \partial_s r \ , \tag{2}$$

where the stress quantity $S = 2\nabla W(C)$ has been introduced. Due to the above assumptions, the Hamiltonian of the system,

$$H(x) = H(r, p, C) = \int_0^L \left(\frac{1}{2\rho A} p \cdot p + W(C) - r \cdot b \right) \mathrm{d}s \ , \tag{3}$$

can be expressed in terms of position r, the momentum density $p = \rho A\dot{r} \in \mathbb{R}^3$ and the strain type quantity $C \in \mathbb{R}$. Taking the balance of linear momentum (1), the kinematic relation between the velocity $v \in \mathbb{R}^3$ and the position r, and the strain rate yields the PH formulation, which consists of the set of PDE of first order in time

$$\begin{bmatrix} \dot{r} \\ \dot{p} \\ \dot{C} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 2\partial_s(\square\,\partial_s r) \\ 0 & 2\partial_s r \cdot \partial_s \square & 0 \end{bmatrix} \begin{bmatrix} -b \\ v \\ \frac{1}{2}S \end{bmatrix} \quad \Leftrightarrow \quad \dot{x} = \mathscr{J}(x)\delta_x H \tag{4}$$

with the formally skew-adjoint operator $\mathscr{J}(x)$ and the (Neumann) boundary conditions

$$u(t) = \begin{bmatrix} -n(0,t) \\ n(L,t) \end{bmatrix}, \tag{5}$$

which define the inputs in the sense of system theory. The power balance

$$\dot{H} = \int_0^L \delta_x H \cdot \dot{x}\,\mathrm{d}s = \int_0^L \delta_x H \cdot (\mathscr{J}(x)\delta_x H)\,\mathrm{d}s = [n \cdot v]_0^L = u \cdot y \tag{6}$$

immediately follows and defines the power-conjugated outputs $y(t) = [v^T(0,t)\ \ v^T(L,t)]^T$.

3 Structure-preserving discretization

By applying a mixed finite element approximation with trial and test functions from the same spaces, the finite-dimensional PH system (where boundary inputs appear as input vectors $\hat{u}$)

$$\begin{bmatrix} \dot{\hat{r}} \\ \dot{\hat{p}} \\ \dot{\hat{C}} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -I & 0 & -\tilde{K}(\hat{r}) \\ 0 & \tilde{K}(\hat{r})^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \nabla_{\hat{r}}\hat{H} \\ \nabla_{\hat{p}}\hat{H} \\ \nabla_{\hat{C}}\hat{H} \end{bmatrix} + \begin{bmatrix} 0 \\ B_{\hat{p}} \\ 0 \end{bmatrix} \hat{u}, \quad \hat{y} = B_{\hat{p}}^T \nabla_{\hat{p}}\hat{H} \quad \Leftrightarrow \quad \dot{\hat{x}} = J(\hat{x})\nabla\hat{H} + B\hat{u}, \quad \hat{y} = B^T\nabla\hat{H}\ , \tag{7}$$

with the skew-symmetric matrix J and the discrete Hamiltonian $\hat{H}(\hat{r},\hat{p},\hat{C})$ is obtained. The discrete power balance, $\dot{\hat{H}} = \hat{y}^T\hat{u}$, defines the canonical discrete output $\hat{y}$ and the passivity of the spatially discretized system follows. A temporal discretization of (7) using discrete gradients $\bar{\nabla}(\square)$ in the sense of Gonzalez [8] yields an energy-momentum consistent time-stepping scheme, i.e.

$$\hat{x}_{k+1} - \hat{x}_k = hJ(\hat{x}_{k+1/2})\bar{\nabla}\hat{H} + Bu_k \tag{8}$$

with piecewise constant inputs.

4 Conclusion

In our talk, we show the PH formulation and structure-preserving discretization of hyperelastic strings using mixed finite elements. The resulting finite-dimensional PH state space model can be used for model order reduction and control design. Besides the solution of the inverse dynamics [7], state estimation and feedback control for the highly underactuated system will offer challenges for future work.

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References

[1] V. Duindam, A. Macchelli, S. Stramigioli, H. Bruyninckx. Modeling and Control of Complex Physical Systems. Springer Berlin Heidelberg, 2009.

[2] R. Rashad, F. Califano, A.J. van der Schaft, S. Stramigioli. Twenty years of distributed port-Hamiltonian systems: A literature review. IMA Journal of Mathematical Control and Information, 37(4), 1400–1422, 2020.

[3] A. Warsewa, M. Böhm, O. Sawodny and C. Tarín. A port-Hamiltonian approach to modeling the structural dynamics of complex systems. Applied Mathematical Modelling, 89, 1528–1546, 2021.

[4] A. Brugnoli, R. Rashad, F. Califano, S. Stramigioli, D. Matignon, Mixed finite elements for port-Hamiltonian models of von-Kármán beams. IFACPapersOnLine, 54(19), 186–191, 2021.

[5] S.S. Antman. Nonlinear Problems of Elasticity. Springer, 2005.

[6] T. Thoma, P. Kotyczka. Port-Hamiltonian FE models for filaments. IFACPapersOnLine 55.30, 353–358, 2022.

[7] T. Ströhle, P. Betsch. A simultaneous space-time discretization approach to the inverse dynamics of geometrically exact strings. International Journal for Numerical Methods in Engineering 123.11, 2573–2609, 2022.

[8] O. Gonzalez. Time integration and discrete Hamiltonian systems. Journal of Nonlinear Science, 6, 449–467, 1996.