Approaching Linear Elastic Deformations of Flexible Bodies via Screw Theory

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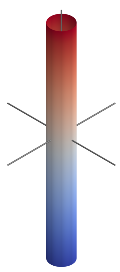
EXTENDED ABSTRACT

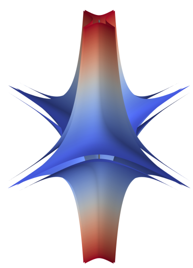
Introduction. The *theory of screws* [2] can nowadays be seen as an established and central tool in the areas of computational kinematics and dynamics of multibody systems, in particular, for the application field of robotics [9, 18, 15]. The geometric element of a screw extends the geometric element of a line [17] and incorporates the rotational and translational aspects of motion and action in one unified concept. In instantaneous kinematics the velocity screw – the twist – entails angular and linear velocity, in instantaneous statics the force screw – the wrench – entails force and torque aspects of effort. Screw theory is applied to identify singularities of mechanisms and to classify them [10, 8]. Furthermore, screw theory allows to consider mechanics of multibody systems also in the context of Lie theory [16] and in the context of geometric algebras [19, 12, 11]. Next to the instantaneous analysis, screw theory is also applicable to compute with relative finite displacements in mechanical systems. The *calculus of motors* [22, 6], the ‘dual-number extension of the Rodrigues-formula’ [13] and the ‘product-of-exponentials formula’ [7] provide fundamental relationships for this purpose.

The usage of compliant mechanisms [14] in the design of robotic systems has become an ongoing trend over the recent years. Deformable materials and soft robots are used for grasping tasks, for collaboration and interaction with the human, for precision applications, and for extreme environments under vacuum, low temperature, or radioactive conditions. Since the theory of screws is based on the assumption of perfect rigid bodies it is not readily applicable to the analysis of flexible systems [3]. The fundamental ‘law of motion’ for flexible bodies is given by Hooke’s law relating the deformation (finite strain) of a flexible body linearly with the force density (stress) acting upon it using the algebraic form of tensors [1, 20].

Objectives, Methods, and Results. With our paper we would like to contribute to a integral view of the ‘displacement domain of rigid bodies’ and the ‘deformation domain of flexible bodies’. In pursuit of this general goal, we consider the geometry of linear spatial deformations of flexible bodies in analogy to the geometry of linear spatial displacements of rigid bodies. Based on this analogy consideration we aim to identify screw-like properties in the domain of linear deformations. Next to the methods given by screw theory and motor calculus, the paper will use the distinction of the three binary number systems [21] of complex numbers, dual numbers, and double numbers that can be recognized in the theory of screws, but also in the contexts of quaternions and their extensions, matrix theory, and geometric algebra [19]. The main working direction is given by applying the *principle of transference* [4], a fundamental principle underlying the theory of screws, to the system of double numbers – instead of to the system of complex numbers. Further, the matrix exponential and the Cayley map are used as crucial tools.



(a) Dual-complex unit circle illustrated of as a *cylinder* in space that indicates the set of all displacement sizes (‘dual angles’). A cylinder along the axis of the Mozzi–Chasles theorem is invariant with respect to any screw displacement about this axis.



(b) Dual-double unit circle illustrated as a *maltese cross* in space that indicates the set of all deformation sizes (‘dual charges’). This shape is the basis for the elastic pendant of the Mozzi–Chasles theorem for oriented-volume preserving deformations.

Figure 1: Visualizations of the dual extensions for the unit circle of the complex-number plane and the double-number plane.

Following this work approach, a dual-number based formalism for modeling linear deformations of flexible bodies is obtained. In particular, the concepts of a *dual angle*, a *spear*, and a *motor* for describing linear finite displacements in terms of screw theory are converted into corresponding concepts for describing linear finite deformations. The obtained concepts are feasible for linear deformations which keep the *volume* and the *orientation* of the material of a flexible body unchanged. In Figure 1, the set of the 'dual arguments' of displacements along a fixed axis is illustrated next to the set of the 'dual arguments' of deformations along a fixed axis. The dual argument of a spatial deformation, the 'dual charge' for short, encodes the amount of *radial shear* and the amount of *axial squeeze* – in analogy to the amount of *rotation* and the amount of *translation* given by the dual argument of a spatial displacement. Based on these concepts, a sibling version of the Mozzi–Chasles theorem about the existence of an invariant axis for linear displacements is stated for the case of linear deformations. Variants of the 'cylindric' Euler–Rodrigues formula for spatial displacements [13, 5] suitable for the case of linear deformation matrices are reported.

Conclusions. The intended paper unveils an interconnection between the methods established in screw theory for treating the kinematics and dynamics of rigid bodies and methods for flexible bodies on the other hand. By employing the principle of transference to double numbers, novel geometric concepts are obtained that characterize the geometry of linear deformations in a similar fashion as screw theory. The work may be used in the future to study the motion laws by Newton and Euler and the motion law by Hooke in close analogy and to model linear spatial reconfigurations – that are combinations of displacements and deformations – of multibody systems within a common language.

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