Numerical integration of quasi-linear hyperbolic PDEs governing the inverse dynamics of flexible mechanical systems

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EXTENDED ABSTRACT

How does a flexible body has to be excited on a given boundary, such that the motion on a disjunct boundary can be prescribed?

This seemingly simple question will be pursued in this contribution and problems, that arise by aiming to solve such inverse dynamics problems of (nonlinear) elasticity using classical numerical solution strategies, will be pointed out. Therefore, we will show that the dynamics of the underlying problem is governed by quasi-linear hyperbolic partial differential equations of the form

$$A\partial_t^2 r - \mathrm{div}_s(B\partial_s r) = C. \tag{1}$$

Quasi-linearity of the problem at hand implies, that the coefficients A , B and C are explicitly allowed to depend on the space and time variables $s \in S \subset \mathbb{R}^\alpha$ and $t \in T = [0, \infty)$ as well as on the solution $r : S \times T = \Omega \subset \mathbb{R}^{\alpha+1} \mapsto \mathbb{R}^d$ itself and its first partial derivatives (see Fig. 1 for an illustration of the space-time domain for $\alpha = 1$ and $\alpha = 2$). That is,

$$A, B : \bar{\Omega} \mapsto \mathbb{R}^{d,d} \qquad \text{and} \qquad C : \bar{\Omega} \mapsto \mathbb{R}^d \qquad \text{where} \qquad \bar{\Omega} = \Omega \cup \{(r, \partial_s r, \partial_t r) : \Omega \mapsto \mathbb{R}^d\}. \tag{2}$$

Since we are interested in the inverse dynamics problem, we ask for the unknown *Neumann*-boundary conditions

$$B\partial_s r(\partial\Omega_f) = f(t). \tag{3}$$

on $\partial\Omega_f = \partial S_f \times T$, such that, some given time-variant *Dirichlet*-boundary conditions on $\partial\Omega_\gamma = \partial S_\gamma \times T$

$$r(\partial\Omega_\gamma) = \gamma(t) \tag{4}$$

are fullfilled. In (3) and (4), the unknown actuation $f(t) : \partial\Omega_f \mapsto \mathbb{R}^d$, acting on $\partial\Omega_f$ and the partly prescribed motion $\gamma(t) : \partial\Omega_\gamma \mapsto \mathbb{R}^d$ on $\partial\Omega_\gamma$, has been introduced, respectively. In case of flexible multibody dynamics, additionally *Neumann* boundary conditions might be imposed on $\partial\Omega_\gamma = \partial S_\gamma \times T$

$$B\partial_s r(\partial\Omega_\gamma) = n(t). \tag{5}$$

Herein $n(t) : \partial\Omega_\gamma \mapsto \mathbb{R}^3$ denotes the contact force acting on $\partial\Omega_\gamma$. Finally, initial conditions

$$r(\partial\Omega_0) = r_0(s) \quad \text{and} \quad \partial_t r(\partial\Omega_0) = v_0(s) \quad \forall \quad s \in S \tag{6}$$

are imposed on $\partial\Omega_0 = S \times \{0\}$.

By applying suitable integration methods in space, the given initial boundary value problem might be transformed into ordinary differential (spatially discrete) equations subjected to servo constraints. In this case, the unknown *Neumann*-boundary conditions (3) could be considered as Lagrange multipliers enforcing the prescribed, time-varying *Dirichlet*-boundary condition.

The motion of such systems (without assuming any further geometric constraints) are governed by the following differential algebraic equation (DAE)

$$\begin{aligned} M\,\mathrm{D}_t^2 q(t) + F(\mathrm{D}_t q(t), q, t) + G^T f(t) &= 0 \\ g(q,t) = Hq(t) - \gamma(t) &= 0. \end{aligned} \tag{7}$$

Herein the nodal configuration vector $q : T \mapsto \mathbb{R}^k$ and the actuating components $f : T \mapsto \mathbb{R}^m$, for $m < k$ have been introduced. We will show that due to the spatially disjunct, hence non-standard construction of the constraint realization, the resulting DAEs are in general characterized by either a high differentiation index or the appearence of (unstable) internal dynamics, depending on the spatial discretization.

In contrast to ideal contact constraints, servo constraints $(7)_2$ in general do not have collocation property. Geometrically this means, that the Lagrange multipliers, enforcing the constraints are not orthogonal to the constraint manifold anymore. This geometrical properties of the constraint realization are specified by

$$n = \mathrm{rank}(HM^{-1}G^T). \tag{8}$$

Three distinct cases of the resulting orientation of the actuation $G^T f(t)$ on the constraint manifold $Q = \{q : g(q,t) = 0\}$ can be identified (cf. e.g. [3]):

(i)	$n = m$	(non-ideal) orthogonal. All m constraint components can be actuated
(ii)	$0 < n < m$	mixed orthogonal-tangential. Only n constraint components can be actuated directly.
(iii)	$n = 0$	tangential. None of the constraint components can be actuated directly.

Orthogonal constraint realizations lead to differentially non-flat systems, where (unstable) internal dynamics may arise. For mixed orthogonal-tangential and fully tangential constraint realizations, the system at hand is possibly differentially flat or non-flat, either without or with internal dynamics. In case of non-flat systems (unstable) inverse dynamics may occure, hindering numerical integration of the problem at hand. Therefore, it is inevitable to carry out relevant analysis thereof (cf. [4] and [5]). On the other hand, flat systems lead to DAEs, that are characterized by high differentiation index. Since a numerically stable solution of the resulting DAEs is depending significantly on the differentiation index, it is inevitable to reduce the index in order to get a stable numerical solution (cf. [1] and references therein). We will demonstrate that both, internal dynamics and high differentiation index DAEs are a direct consequence of the discretization process. This obviously restricts the applicability of the classical semi-discretization approaches.

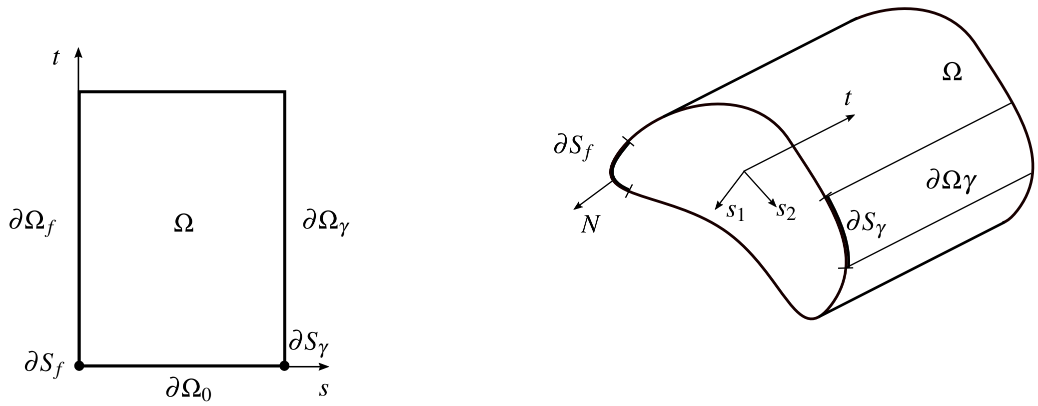

Figure 1: Space-time domain for $\alpha = 1$ (left) and $\alpha = 2$ (right).

Therefore, we aim to analyse the initial boudary value problem in more detail, i.e. exposing the underlying hyperbolic structure of the governing partial differential equations anticipates to gain more insights into the problem at hand. By enlighting resulting mechanisms such as wave propagation, it will become more and more apparent, that a simultaneous space-time integration is essential to solve the inverse dynamics problem numerically stable (cf. [2]). In fact, it will be demonstrated, that the sequential discretization leads to incomplete data on the boundary that causes an ill-posedness of the problem at hand. Motivated by this new insights, we will be able to present novel numerical methods based on simultaneous space-time integration of the initial boundary value problem at hand.

Numerical examples, underpin the relevance of the presented methods.

References

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