The geometrically exact beam model with a normalized quaternion discretization

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EXTENDED ABSTRACT

1 Introduction

Beam models play an important role in the efficient simulation of slender structures in many different fields of engineering. The most important beam model for large displacements is the so-called geometrically exact beam also often referred to as Simo-Reisner beam [1]. The configuration manifold of the beam model is given by $R^3 \times SO(3)$ as it describes the position of the centerline as well as the orientation of the beam's cross-section. The partial differential equation describing the behavior of the beam is usually solved with the help of the Finite Element Method (FEM). So it becomes necessary to discretize the special orthogonal group $SO(3)$ in a finite element sense.
A finite element discretization of the special orthogonal group is rather difficult as $SO(3)$ is not an abelian, additive group but a matrix group under multiplication [2]. Though there exist parametrizations of $SO(3)$, which have an additive structure, they exhibit singularities and result in a complex formulation. This can be overcome by discretizing the $SO(3)$ directly by using so-called directors [3]. The directors can be discretized additively, so in a classical finite element sense. This however leads to an increase in the number of degrees of freedom, as it becomes necessary to introduce Lagrange multipliers to ensure the orthonormality of the directors. The use of unit quaternions for the parametrization presents an interesting alternative [4]. Even though unit quaternions have a complex mathematical structure, their unit length can be assured using a normalization technique, while still applying an additive discretization technique.
In the literature, it is often shown that the Isogeometric Analysis (IGA) is advantageous over the classical FEM with Lagrangian elements, especially for dynamic problems. We thus apply the IGA to the quaternion formulation of the geometrically exact beam.

2 Geometrically exact beam

A slender structure, which is extended much longer in one dimension than in the other two, can be modeled using a beam model.

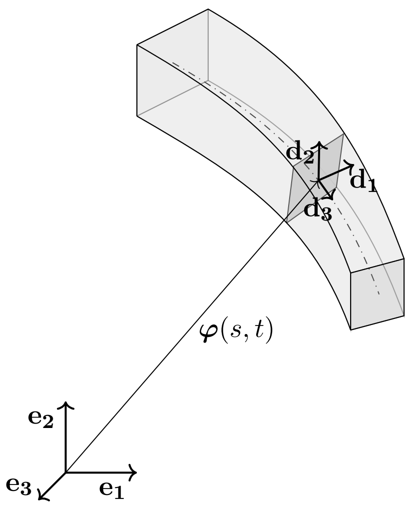

Figure 1: Sketch of the geometrically exact beam

The configuration of the geometrically exact beam model is described by the position of its centerline $\varphi(s,t)$ and the orientation of its cross-section plane spanned by the vectors $\mathrm{d}_\alpha(s,t)$. The coordinate $s \in [0,1]$ is the parametrization of the centerline and t is the time. Thus, every point on the beam can be described by

$$\mathrm{x}(s,t) = \varphi + \theta^\alpha \mathrm{d}_\alpha \tag{1}$$

with θ_α the convective coordinates. Note that we use the Einstein notation for double indices. In this work Greek letter indices run from α, β = 1-2 and Roman letter indices from i, j, k = 1-3. Together with a third director d_3

$$\mathrm{d}_3(s,t) = \mathrm{d}_1(s,t) \times \mathrm{d}_2(s,t) \tag{2}$$

the directors d_i form an orthonormal frame. The directors are of unit length and mutually orthogonal

$$\mathrm{d}_i(s,t) \cdot \mathrm{d}_j(s,t) = \delta_{ij} \tag{3}$$

We thus can write the configuration $\mathbb{Q}$ of the geometrically exact beam as

$$\mathbb{Q} = \{(\varphi, \mathrm{d}_i) : [0,L] \times [0,T] \to \mathbb{R}^3 \times \mathbb{R}^{3\times 3} : \mathrm{d}_i \cdot \mathrm{d}_j = \delta_{ij}\} \tag{4}$$

Another way to express the directors is to use a special orthogonal tensor $\mathrm{R}(s,t)$ and rotate the orthonormal basis e_i

$$\mathrm{d}_i(s,t) = \mathrm{R}(s,t) \cdot \mathrm{e}_i \tag{5}$$

Here it becomes necessary to parametrize $\mathrm{R}(s,t)$ in some sense. Instead of a rotation vector, which leads to singularities, we chose a parametrization via unit quaternions [5].
With the help of unit quaternions, we can express the beams configurations as

$$\mathbb{Q}_\mathrm{q} = \{(\varphi, \mathrm{p}_i) : [0,L] \times [0,T] \to \mathbb{R}^3 \times \mathbb{R}^4 : \mathrm{p}_i \cdot \mathrm{p}_i = 1\} \tag{6}$$

Unit quaternions form a Lie group under the quaternion product. This has to be considered when discretizing quaternions with finite elements. The use of a classical additive discretization procedure would not result in an exact conservation of the underlying manifold S^3 at every point.
Another possibility is to use a multiplicative discretization. This way the S^3 structure is conserved exactly. However, when using higher discretization orders the computation of derivatives becomes rather cumbersome [6].
We thus propose a more convenient approach with a normalization projector. The discretized quaternion $\mathrm{q}_h(s)$ then reads as

$$\mathrm{q}_h(s) = \mathrm{P}(\mathrm{p}_h) = \frac{\sum_{i=1}^n N_i(s)\mathrm{p}_i}{|\sum_{i=1}^n N_i(s)\mathrm{p}_i|} \tag{7}$$

where p_i are the nodal values of the quaternions and $N_i(s)$ are the corresponding shape functions. p_h is obtained through the usual finite element discretization $\mathrm{p}_h = \sum_{i=1}^n N_i(s)\mathrm{p}_i$. The shape function can either be classical Lagrangian shape functions or NURBS in the IGA framework. Through the use of the normalization projector P we ensure a unit length of the quaternions $\mathrm{q}_h(s)$ at every point s inside the discretization domain. Thus, at every point $\mathrm{q}_h(s)$ is a parametrization of S^3 and consecutively of $SO(3)$. Eq. (7) thus presents a relatively simple discretization procedure for the complex structure of $SO(3)$.

3 Conclusion

The configuration of the geometrically exact beam has a complex mathematical structure as it involves rotations, which are not additive in three dimensions. Thus great care has to be taken when applying a discretization technique such as the FEM to preserve the underlying structure of the rotation. We propose a new relatively simple discretization approach for a quaternion description of the beam.

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