Non-Smooth Dynamics Formulation for Planar Translational Joints with Clearances

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1 Introduction and Motivation

Multibody systems can be classified into two categories based on the type of constraints they carry. Ideal constraints in the multibody systems enforce absolute alignment of bodies with respect to each other in the desired direction of motion. However, the real-life systems have bodies constrained to each other in joints which have clearances, and the condition of absolute alignment is not followed. The existing multibody formulations deal with such cases by introducing impulse generating contact detection models based on interpenetration of the bodies, which estimate the contact forces using material properties of the bodies in contact using nonlinear spring-damper elements [1]. The high numerical values of Hertzian contact stiffness result in stiff differential equations. However, these models do not include the mathematical representations of such constraints with clearances. The constraints having clearances are inequality expressions. In such a case, the degrees of freedom of the constrained bodies cannot be time-invariant and are event based. The bodies may form contact points which may change with time as the bodies change orientation. Hence, study of such a system could result as of non-smooth nature. This work delineates a methodology for formulating and simulating non-smooth planar multibody systems with an example of translational joints.

2 Theoretical foundation and model formulation

Haug [2] derived the equations of motion for unconstrained systems by leveraging D'Alembert's principle of virtual work:

$$\delta\mathbf{q}(\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{S}(\mathbf{q},\dot{\mathbf{q}}) - \mathbf{Q_A}) = \mathbf{0} \tag{1}$$

Here, $\mathbf{q} \in \mathbb{R}^3$ is the vector of generalized coordinates and $\delta\mathbf{q}$ represents the vector of virtual displacements. The entities $\mathbf{M}(\mathbf{q})$, $\mathbf{S}(\mathbf{q},\dot{\mathbf{q}})$ and $\mathbf{Q_A}$ respectively, represent the mass matrix, the vector of Coriolis forces and the vector of externally applied forces and torques. Unless the contact happens, the bodies move as an unconstrained system in the free space, following equation (2):

$$\ddot{\mathbf{q}} = \mathbf{M}(\mathbf{q})^{-1}(\mathbf{Q_A}(\boldsymbol{\mu},\mathbf{q},\dot{\mathbf{q}}) + \mathbf{S}(\mathbf{q},\dot{\mathbf{q}})) \tag{2}$$

If the contact happens, the work done by constraint reaction forces corresponding inequality constraints, is 0. Therefore, $\delta\mathbf{q}\left(\boldsymbol{\Phi}_{\mathbf{q}}^{\mathbf{ineq}}(\mathbf{q})\right) = \mathbf{0}.$ Using the Lagrangian multipliers [3], the equation of motion becomes:

$$\delta\mathbf{q}(\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{S}(\mathbf{q},\dot{\mathbf{q}}) - \mathbf{Q_A}) + \delta\mathbf{q}\left(\boldsymbol{\Phi}_{\mathbf{q}}^{\mathbf{ineq}}\right)^{\mathbf{T}}\boldsymbol{\mu} = \mathbf{0} \tag{3}$$

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \left(\boldsymbol{\Phi}_{\mathbf{q}}^{\mathbf{ineq}}\right)^{\mathbf{T}}\boldsymbol{\mu} - \mathbf{S}(\mathbf{q},\dot{\mathbf{q}}) - \mathbf{Q_A} = \mathbf{0} \tag{4}$$

Where $\boldsymbol{\mu}$ is the vector of Lagrange multipliers for each constraint equation. The entity $(\boldsymbol{\Phi}^{\mathbf{ineq}}).*\,\boldsymbol{\mu}$ represents the element wise product of expressions in the constraint manifold $\boldsymbol{\Phi}^{\mathbf{ineq}}$, and the corresponding Lagrange multiplier $\boldsymbol{\mu}$. Therefore:

$$(\boldsymbol{\Phi}^{\mathbf{ineq}}(\mathbf{q})).*\,\boldsymbol{\mu} = \mathbf{0}, \text{where } \boldsymbol{\mu} \geq \mathbf{0} \tag{5}$$

This equation indicates a complementary condition which basically means if $(\boldsymbol{\Phi}^{\mathbf{ineq}}(\mathbf{q})) = \mathbf{0}$, $\boldsymbol{\mu} \neq \mathbf{0}$ and if $\boldsymbol{\mu} = \mathbf{0}$, $(\boldsymbol{\Phi}^{\mathbf{ineq}}(\mathbf{q})) \neq \mathbf{0}$. The physical meaning of this equation is that if the contact happens, i.e., $\boldsymbol{\Phi}^{\mathbf{ineq}}(\mathbf{q}) = \mathbf{0}$, $\boldsymbol{\mu}$ should not be 0 as it will give out contact forces at each of the contact points. Similarly, if $\left(\boldsymbol{\Phi}_{\mathbf{q}}^{\mathbf{ineq}}\right) \neq \mathbf{0}$, this means that the contact doesn't happen and corresponding $\boldsymbol{\mu} = \mathbf{0}$. Further, using Gauss' principal of least squares and least action [4], the problem can be stated as a quadratic programming problem subjected to constraints with objective function to be minimized as given below:

$$\text{Minimize } \left(\mathbf{M}(\mathbf{q})^{-1}\left(\boldsymbol{\Phi}_{\mathbf{q}}^{\mathbf{ineq}}\right)^{\mathbf{T}}\boldsymbol{\mu}\right)^{\mathbf{T}}\mathbf{M}(\mathbf{q})\left(\mathbf{M}(\mathbf{q})^{-1}\left(\boldsymbol{\Phi}_{\mathbf{q}}^{\mathbf{ineq}}\right)^{\mathbf{T}}\boldsymbol{\mu}\right) \tag{6}$$

Subjected to: $\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \left(\boldsymbol{\Phi}_{\mathbf{q}}^{\mathbf{ineq}}\right)^{\mathbf{T}}\boldsymbol{\mu} + \mathbf{Q_A}(\boldsymbol{\mu},\mathbf{q},\dot{\mathbf{q}}) + \mathbf{S}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{0}$; $\boldsymbol{\Phi}^{\mathbf{ineq}}(\mathbf{q})_{\mathbf{q}}\dot{\mathbf{q}} \leq 0$; $(\boldsymbol{\Phi}^{\mathbf{ineq}}(\mathbf{q})).*\,\boldsymbol{\mu} = \mathbf{0}$; and $\boldsymbol{\mu} \geq \mathbf{0}.$

3 Case study of a planar translational joint

Figure 1 shows a simple case study to illustrate the proof of concept for the proposed formulation. The case study comprises of a slider with its center of mass denoted by the generalized coordinates $\mathbf{q} \in \mathbb{R}^{\mathbf{3}}$and a massless rod fixed at both ends. There is a clearance $c = 10\mu m$ (0.01mm) between the slider and the rod and gravity acts along the **-Y** axis. An external force $\mathbf{F} \in \mathbb{R}^{\mathbf{3}}$ acts on the slider at a point marked by a body-frame vector $\mathbf{s}_{\mathbf{f}}'$.

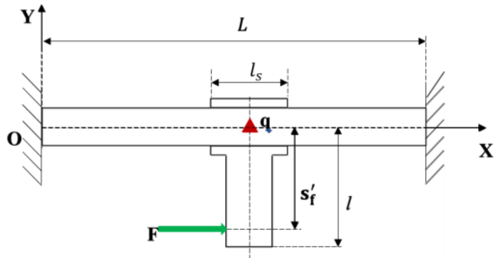

Figure 1: Schematic diagram of the case study

Keeping the inertial properties and applied external force constant, the following three cases are possible depending on the length of $\mathbf{s}_\mathbf{f}'$ denoted by $||\mathbf{s}_\mathbf{f}'||$: **(i)** If $||\mathbf{s}_\mathbf{f}'|| = 0$, i.e., the force **F** acts on the center of mass **q**, and gravity acting along the **-Y** axis. **(ii)** If $||\mathbf{s}_\mathbf{f}'||$ is a small value relative to the full length of the protrusion l, the torque applied by gravity, acting along the **-Y** axis, about any contact point could surpass the torque generated by the applied force **F**. **(iii)** If $||\mathbf{s}_\mathbf{f}'|| = l$ with gravity acting along the **-Y** axis. All three possible cases must be studied to validate the formulation for different types and locations of possible contacts.

4 Preliminary results

For case 1 and case 3 discussed above, the position vs. time plots of the centroidal Y coordinates along with the Y coordinates of left-end and right-end of the slider lying on the central axis are shown in Figure 2 (*a*) and Figure 2 (*b*) respectively. The simulation results were achieved with a constant time step size of 10^{-3} seconds with constraint violation $\left|\left|\mathbf{\Phi}^{\mathbf{ineq}}(\mathbf{q})\right|\right| \leq 10^{-10}$ upon the contact formation. As it is expected in case 2, also demonstrated in Figure 2 (*a*), in absence of the applied external torque, the slider would retain its initial angular orientation. However, gravity pulls the slider downwards by 0.01mm (value of clearance) to make a line contact with the rod.

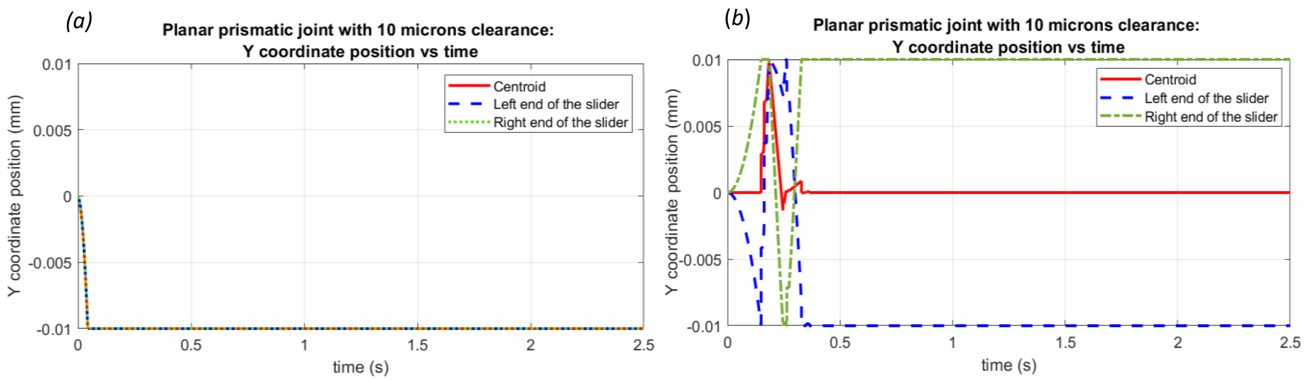

Figure 2: Y coordinates vs. time: (*a*) Line contact case, (*b*) Two points in contact

In case 3, as shown in Figure 2 (*b*), the slider changes its angular orientation until point contact is detected at each end of the slider. The right end of the slider moves upwards by 0.01mm, while the left end descends downwards by 0.01mm and thus the contact is formed with the rod at end points situated diagonally opposite with respect to each other. After contact is formed, the slider retains its angular orientation and moves along **X** direction without losing contact.

5 Conclusions

The simulation results demonstrate the efficiency of the non-smooth dynamics methodology for the case study with line contact and constant two-point contact. The mathematical representations of constraint inequalities are satisfied within significant accuracy. The computational time for the popular continuous contact models is relatively higher as these models evaluate contact impulse using nonlinear spring-damper elements as functions of interpenetration between the bodies. The resulting high stiffness of the differential equation often requires small time-step sizes during contact force evaluation. This makes the continuous contact models computationally expensive, especially when the contacts break and form intermittently. The study targets to investigate the same for the case where the torque generated by external force **F** is less than the torque generated by gravity about the contact point. In such a case, the simulation is expected to be able to capture the formation and breaking of contact. The further scope of this work includes studying planar revolute joints with clearance and then extending the methodology for spatial systems.

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