A Sparse Constrained Krylov-Schur Eigenvalue Solver for the Aeroelastic Stability Assessment of Multi-Flexible-Body Systems

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EXTENDED ABSTRACT

1 Introduction

Modern wind turbine has encountered the risk of fatigue failure due to the extremely low and even negative aeroelastic damping in the rotor edgewise modes. The modal damping ratio of wind turbine can be extracted from the eigenvalues of linearised equation of motion. Besides the stability assessment of dynamic systems, the eigenvalue problem also presents in the component mode synthesis and other applications. The equation of motion of constrained multi-flexible-body systems consists of large but sparse matrices. For the sake of acceptable computational performance, the eigensolver should preserve the sparsity of the matrices, and should be able to output just a small subset of eigenvalues, either the lowest ones or those clustered around a frequency of interest. Krylov-Schur method has been presented in [1] as an improvement over previous Krylov subspace methods such as Lanczos, Arnoldi or IRAM, offering superior robustness and faster convergence [2]. It has become the default method for MATLAB `eigs` command, and it is also available in the SLEPc and TRILINOS libraries. However both are large libraries that target supercomputing and require complex build toolchains.

2 Problem description

We consider the Differential Algebraic Equations (DAE) of a generic, nonlinear multi-flexible body system with generalized coordinates $\boldsymbol{q} \in \mathbb{R}^n$ and holonomic-rheonomic constraints $\boldsymbol{C}(\boldsymbol{q},t) = \boldsymbol{0}$ with a $m \times n$ sparse jacobian $C_q(\boldsymbol{q},t) = \frac{\partial \boldsymbol{C}(\boldsymbol{q},t)}{\partial \boldsymbol{q}}$:

$$\begin{cases} M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C_q(\boldsymbol{q},t)^T \boldsymbol{\gamma} = \boldsymbol{f}(\boldsymbol{q},\dot{\boldsymbol{q}},t) - \boldsymbol{f}_g(\dot{\boldsymbol{q}}) & (1) \\ \boldsymbol{C}(\boldsymbol{q},t) = \boldsymbol{0} & (2) \end{cases}$$

Here $\boldsymbol{f}$ are external and internal forces, $\boldsymbol{f}_g$ are gyroscopic forces. After linearization about the dynamic equilibrium point, introducing $R(\boldsymbol{q},\dot{\boldsymbol{q}})$ and $K(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},\boldsymbol{\gamma})$ stiffness and damping matrices:

$$\begin{cases} M(\boldsymbol{q})\delta\ddot{\boldsymbol{q}} + R(\boldsymbol{q},\dot{\boldsymbol{q}})\delta\dot{\boldsymbol{q}} + K(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},\boldsymbol{\gamma})\delta\boldsymbol{q} + C_q(\boldsymbol{q},t)^T \delta\boldsymbol{\gamma} = \boldsymbol{0} & (3) \\ C_q(\boldsymbol{q},t)\delta\boldsymbol{q} = \boldsymbol{0} & (4) \end{cases}$$

$$R(\boldsymbol{q},\dot{\boldsymbol{q}}) = -\frac{\partial \boldsymbol{f}(\boldsymbol{q},\dot{\boldsymbol{q}},t)}{\partial \dot{\boldsymbol{q}}} + \frac{\partial \boldsymbol{f}_g(\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}, \qquad K(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},\boldsymbol{\gamma}) = \frac{\partial (M(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{f}_g)}{\partial \boldsymbol{q}} + \frac{\partial (C_q(\boldsymbol{q},t)^T \boldsymbol{\gamma})}{\partial \boldsymbol{q}} - \frac{\partial \boldsymbol{f}(\boldsymbol{q},\dot{\boldsymbol{q}},t)}{\partial \boldsymbol{q}} \tag{5}$$

The conventional Standard Eigenvalue Problem (SEP) of a not damped, linearized system with M,K matrices is $\left(M^{-1}K - \lambda_i\right)\boldsymbol{\Phi}_i$ for eigenvalues $\lambda_i = \omega_i^2$. A way to introduce constraints is to express the following *Generalized Eigenvalue Problem* (GEP)

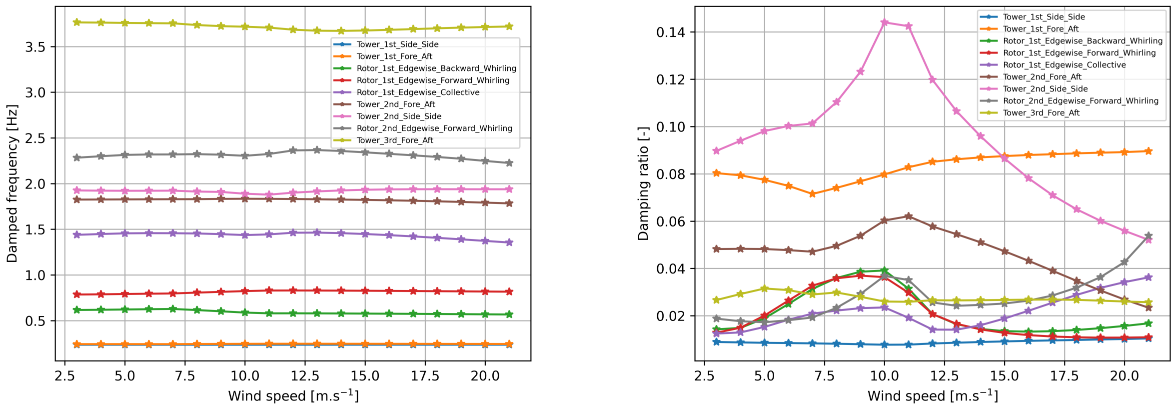
$$-\begin{bmatrix} K & C_q^T \\ C_q & 0 \end{bmatrix} \hat{\boldsymbol{\Phi}}_i = \lambda_i \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \hat{\boldsymbol{\Phi}}_i \tag{6}$$

where we introduced an augmented eigenvector $\hat{\boldsymbol{\Phi}}_i = \{\boldsymbol{\Phi}_i^T, \boldsymbol{\xi}_i^T\}^T$, with a part $\boldsymbol{\xi}_i \in \mathbb{R}^m$ that can be discarded after the computation. This produces m spurious modes with $\lambda_i = \pm\infty$ that correspond to the m constraints and that can be easily filtered.

For the damped case we do a constrained *Quadratic Eigenvalue Problem* (QEP): we assume solutions $\boldsymbol{q} = \boldsymbol{\Phi} e^{\lambda t}$ in (3)(4), we introduce the augmented eigenvector $\underline{\hat{\boldsymbol{\Phi}}}_i \in \mathbb{R}^{2n+m}$ as $\underline{\hat{\boldsymbol{\Phi}}}_i^T = \{\boldsymbol{\Phi}_i^T, \lambda_i \boldsymbol{\Phi}_i^T, \boldsymbol{\xi}_i^T\}$ and we write the QEP as a constrained GEP:

$$\begin{bmatrix} 0 & I & 0 \\ -K & -R & -C_q^T \\ -C_q & 0 & 0 \end{bmatrix} \underline{\hat{\boldsymbol{\Phi}}}_i = \lambda_i \begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{\hat{\boldsymbol{\Phi}}}_i \tag{7}$$

Both cases (6) and (7) cornespond to GEP of the type $A\boldsymbol{\Phi}_i = \lambda_i B\boldsymbol{\Phi}_i$, with the difference that in (6) both A,B are real symmetric, hence with real eigenvalues, whereas in (7) A,B are real and unsymmetric, hence complex eigenvalues are expected in conjugate pairs. Also, despite one could convert the GEP $A\boldsymbol{\Phi}_i = \lambda_i B\boldsymbol{\Phi}_i$ into the SEP $C\boldsymbol{\Phi}_i = \lambda_i \boldsymbol{\Phi}_i$ with $C = B^{-1}A$, this would destroy the sparsity of the A,B matrices: sparsity is a feature to exploit in case of large multibody systems. For this reason we need an eigenvalue solver that can operate directly on GEPs and not only on SEPs. Also, for the above mentioned reasons, it must be able to compute complex-valued eigenpairs.



(a) *Aeroelastic modal frequencies.* (b) *Aeroelastic damping ratios.*

Figure 1: Modes of IEA Wind 15MW reference wind turbine

3 The Krylov-Schur solver

When the degrees of freedom of multi-flexible-body system reach up to the order of thousand, it is difficult to obtain all eigenmodes. In engineering practice, one is always interested in a small subset of them with lowest eigenvalues. This is possible using Krylov subspace methods: we use the Krylov-Schur iteration to this end [1]. The *shift and invert* technique is required to search the eigenvalues λ_i with smallest magnitude, by means of a Moebius transformation:

$$(C-\mu_i I)\hat{\underline{\mathbf{\Phi}}}_i = 0, \quad C = (A-\sigma B)^{-1}B, \quad \lambda = \frac{1}{\mu}+\sigma \tag{8}$$

We experienced that the shift value $\sigma = 1\times 10^{-3}$ works well to extract the lower order modes including the rigid body modes and cure ill-conditioned problems. The exact inversion of $(A-\sigma B)$ is avoided and its sparsity is preserved via factorizing it once at the beginning of the Krylov-Schur iterations, and solving the linear equation $\boldsymbol{r} = C\boldsymbol{v}$ by doing two steps: sparse-multiplication $\boldsymbol{z} = B\boldsymbol{v}$ followed by $\boldsymbol{r} = (A-\sigma B)^{-1}\boldsymbol{z}$. An important improvement on the robustness of the method is achieved by a trivial and inexpensive pre-conditioning of the Jacobian matrix C_q by a scaling trace$(K)/n$. We implemented the method in the open-source multibody library CHRONO[3].

4 Application to the stability assessment

A blade designed with thin airfoil cross sections and made of composite materials exhibits complicated anisotropic characteristics. The *Blade Element Momementum* (BEM) theory is used to evaluate the aerodynamic loadings on the blades, which are then linearised by numerical perturbation to derive the aerodynamic stiffness and damping matrices. The linearised equation of motion of wind turbine aeroelastic coupling system is time variant due to the periodical rotation of rotor: a time invariant system expressed in the inertial frame is obtained via the *Multi-Blade Coordinate* (MBC) transformation $\boldsymbol{q} = \Theta(t)\boldsymbol{y}$. Eigenvalues are solved using the implemented Krylov-Schur iterative solver. Taking the IEA Wind 15MW reference wind turbine model as example, a series of operational status at different wind speeds are analysed. The modal frequencies and damping ratios are extracted, and the less damped modes are plotted on the Campbell diagram. The lowest damping ratio 0.0078 is positive, which indicates the IEA Wind 15MW reference wind turbine is stable.

5 Conclusion

We introduced a sparsity-preserving Krylov-Schur solver that targets the eigenvalue problem with highest generality. The test performed on the aeroelastic stability assessment of a modern wind turbine demonstrates that the proposed solver can effectively handle the case of large scale constrained multi-flexible-body system.

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