Impact of shear and extensional stiffness on equilibrium configurations of elastic Cosserat rods

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EXTENDED ABSTRACT

1 Introduction

Geometrically exact rod models [1] occur in three different variants w.r.t. the kinematical properties of their configuration variables (see Fig. 2 and [2]): *(i)* inextensible Kirchhoff rods, *(ii)* extensible Kirchhoff rods, and *(iii)* Cosserat rods.

We are interested in industrial applications where large spatial deformations of cables have to be simulated interactively [3]. Typical boundary conditions lead to deformed configurations showing a considerable amount of bending of the centerline, accompanied by a moderate amount of approximately uniform twisting of the cross sections along the configuration. In such cases, configurations computed by either of the three model variants turn out to be practically the same, as illustrated by Fig. 1:

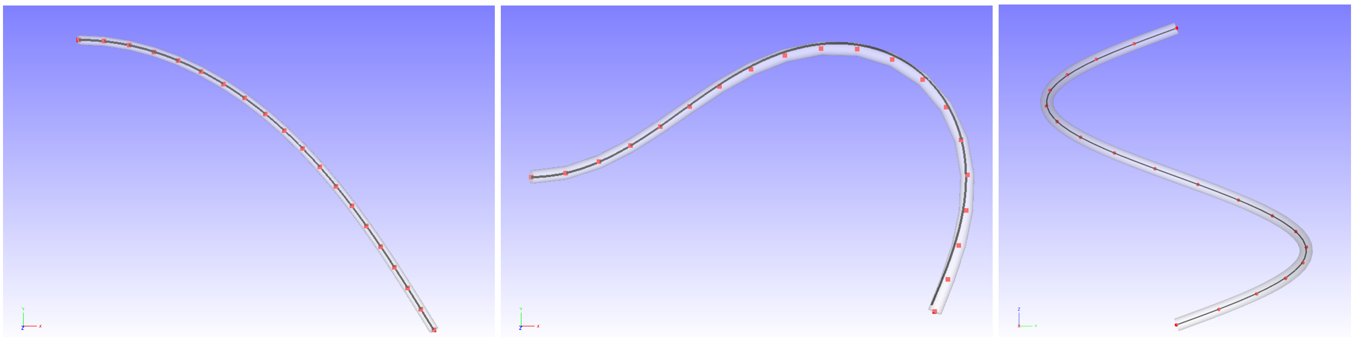

Figure 1: Analytical centerline curves of an inextensible Kirchhoff rod (solid lines) in plane bending (*left:* cantilever type, *middle:* both ends clamped) and helical (*right*) configurations. The red dots show the vertex positions computed with a discrete Cosserat rod model by minimization of the elastic energy (see [4] for details).

In our contribution, we investigate the influence of the effective stiffness parameters $[EA]$ and $[GA]$ of a Cosserat rod model that govern extension (or compression) of the centerline as well as transverse shearing of the cross sections on its equilibrium configurations in such cases. Our results open up the possibility to *set* the stiffness parameters $[EA]$ and $[GA]$ to proper values by *modeling* rather than measurements.

2 Modelling details and methodical approach

For model variants *(i)* and *(ii)* one postulates that the moving frames $\mathsf{R}(s)$ remain adapted to the centerline also in deformed configurations, such that the unit tangent vector $\mathbf{t}(s) = \mathbf{r}'(s)/\|\mathbf{r}'(s)\|$ stays always orthogonal to the frame directors $\mathbf{a}^{(1,2)}(s)$ spanning the local cross section plane, and transverse shearing of the cross sections is kinematically inhibited. Variant *(i)* additionally postulates an inexensible centerline by requiring that $\|\mathbf{r}'(s)\| = 1$ holds for all deformed configurations. Differently, for variant *(iii)* neither adapted frames nor an inextensible centerline are assumed for deformed configurations.

Bending and twisting are affected by the related effective stiffness parameters $[EI]$ and $[GJ]$, respectively. For composite cables, one needs to treat these stiffness parameters as independent quantitites. Often the mass per length ρ_L of a cable is sufficiently low, such that the influence of gravity can be considered as weak, and the shape of deformed configurations in static equilibrium mainly depends on the ratio $[GJ]/[EI]$. Therefore it is important to measure these stiffness parameters properly [3, 5].

While for homogeneous elastic specimens the measurement of extensional stiffness $[EA]$ is an elementary experimental task, obtaining reproducible results from measurements of composite cables turns out to be far more difficult [5], and a measurement of the shear stiffness $[GA]$ is practically impossible. Therefore it is important to understand the influence of the effective stiffness parameters $[EA]$ and $[GA]$ on the rod configurations in equilibrium both qualitatively and quantitatively. Apart from the overall shape, estimates of the extensional strain $\varepsilon_t(s) := \|\mathbf{r}'(s)\| - 1$ and shear angle $\vartheta_s(s) := \arccos(\langle \mathbf{a}^{(3)}(s), \mathbf{t}(s)\rangle)$ are of interest.

Assuming for simplicity zero gravity, equilibrium configurations of a straight inextensible Kirchhoff rod are local minima of its elastic bending and torsional energy, which in the case of transversally isotropic bending stiffness $[EI]$, torsional stiffness $[GJ]$ and

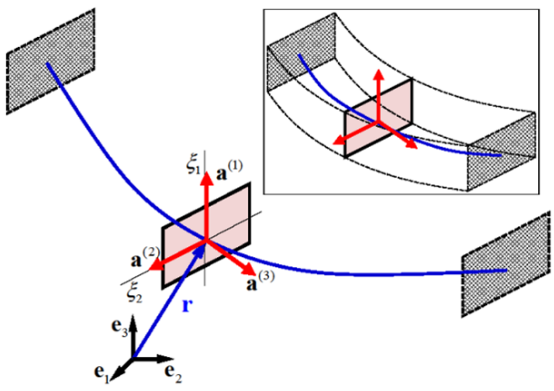

Figure 2: Centerline $\mathbf{r}(s)$ and moving frame $\mathsf{R}(s) = \mathbf{a}^{(k)}(s) \otimes \mathbf{e}_k$ of a Cosserat rod (see [2] for technical details and notation).

cross sections with coinciding shear and area centers is given by $\mathscr{W}_{bt} = \frac{1}{2}\int_0^L ds\,[EI]\varkappa^2 + [GJ]\tau^2$, where $\varkappa(s)$ is the Frenet curvature of the centerline, and $\tau(s)$ is the twist of the adapted frame. For extensible Kirchhoff rods, the total elastic energy consists of the sum $\mathscr{W}_{el} = \mathscr{W}_{bt} + \mathscr{W}_{ext}$ with the extensional energy $\mathscr{W}_{ext} = \frac{1}{2}\int_0^L ds\,[EA]\varepsilon_t^2$. For Cosserat rods, the latter is replaced by the more complex energy term $\mathscr{W}_{es} = \frac{1}{2}\int_0^L ds\,[EA](\Gamma^{(3)} - 1)^2 + [GA](\Gamma^{(1)2} + \Gamma^{(2)2})$ measuring the elastic energy stored in extension and transverse shearing, where the strains $\sqrt{\Gamma^{(1)2} + \Gamma^{(2)2}} = (1+\varepsilon_t)\sin(|\vartheta_s|) \approx |\vartheta_s|$ and $\Gamma^{(3)} = (1+\varepsilon_t)\cos(\vartheta_s) \approx 1 + \varepsilon_t$ depend on both deformation modes in a combined manner.

Introducing L as the characteristic unit to measure length induces $1/L$ as characteristic unit for curvature. This can be utilized to scale the elastic energy terms to dimensionless form as $\bar{\mathscr{W}}_{bt} = \mathscr{W}_{bt}/W_{bt}^0$ and $\bar{\mathscr{W}}_{es} = \mathscr{W}_{es}/W_{es}^0$, with charateristic energy values $W_{bt}^0 := [EI]/L$ and $W_{es}^0 := [EA]L$. For boundary value problems leading to deformed configurations dominated by bending and torsion, one needs to find local minima of the scaled elastic energy $\bar{\mathscr{W}}_{el} = \mathscr{W}_{el}/W_{bt}^0 = \bar{\mathscr{W}}_{bt} + \bar{\mathscr{W}}_{es}/\lambda_0^2$, where $\lambda_0^2 := [EI]/([EA]L^2)$ is a small dimensionless parameter that can be estimated as $\lambda_0 \simeq \mathscr{O}(r_{IA}/L) \ll 1$ for the effective cross section radius $r_{IA} = 2\sqrt{I/A}$. On this basis, one may treat the energy minimization problem by Berdichevsky's approach of variational asymptotic analysis [7] to find approximate solutions for the Cosserat rod model, which to leading order coincide with those of the inextensible Kirchhoff model. For equilibrium values of both the extensional strains and shear angles one finds the order of magnitude estimates $\varepsilon_t, |\vartheta_s| \sim \mathscr{O}(\lambda_0^2)$. For the relative size of the elastic energy terms in equilibrium one finds the estimate $\mathscr{W}_{es} \simeq \lambda_0^2 \mathscr{W}_{bt}$, such that $\mathscr{W}_{el} \simeq (1+\lambda_0^2)\mathscr{W}_{bt}$ holds. The same estimates can be obtained by applying similar arguments as brought forward by Audoly and Pomeau [6] (see section 3.7). We elaborate on the latter by considering the first integral $\mathbf{m} + \mathbf{r} \times \mathbf{f}$ of the equilibrium equations $\mathbf{f}' = \mathbf{0}$, $\mathbf{m}' + \mathbf{r}' \times \mathbf{f} = \mathbf{0}$ which hold for all rod model variants independent of the kinematical constraints.

In our contribution, we outline the technical details of our theoretical approach, and illustrate our findings by typical results of numerical experiments.

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