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Approaching Linear Elastic Deformations of Flexible Bodies via Screw Theory

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ABSTRACT

The theory of screws can be seen as a central tool for modeling the kinematics and dynamics of spatial mechanisms, in particular robotic systems. Since screw theory is based on the assumption of perfect rigid bodies, it is not readily applicable to the analysis of flexible systems. This paper intends to contribute to an integral view of 'displacements of rigid bodies' and 'deformations of flexible bodies' by considering the geometry of linear spatial deformations in analogy to the geometry of linear spatial displacements. For this purpose, the principle of transference is applied to the system of double numbers. Thus, a dual-number based formalism for modeling linear deformations of flexible bodies is proposed including sibling versions of the Mozzi–Chasles theorem and the Euler–Rodrigues formula.

Keywords: theory of screws, calculus of motors, principle of transference, flexible bodies, linear elasticity

1 Introduction

The *theory of screws* [2] and the *calculus of motors* [31] take a central place in the areas of computational kinematics and dynamics of multibody systems, in particular, for the application field of robotics [13, 26, 22]. Here the notion of a screw extends the geometric element of a line [24] to incorporate the rotational and translational aspects of motion (twist) and action (wrench) in one unified concept. Screw theory is applied to identify and classify singularities of mechanisms [16, 9] and to describe the mechanics of multibody systems in the context of Lie theory [23] and geometric algebras [27, 19, 17]. Besides instantaneous analysis, screw theory is also applicable to deal with finite displacements in mechanical systems. The *calculus of motors* [31, 7], the 'dual-number extension of the Rodrigues-formula' [20] and the 'product-of-exponentials formula' [8] provide crucial relationships for this purpose.

The usage of compliant mechanisms [21, 3, 34] in the design of robotic systems has become an ongoing trend over the recent years. In soft robotics, deformable structures are used for grasping tasks and for collaboration and interaction with the human. In precision applications, compliant mechanisms are used in extreme environments under vacuum, low temperature, or radioactive conditions. Since the theory of screws is based on the assumption of perfect rigid bodies, it is not readily applicable to the analysis of flexible systems. The general law of motion for flexible bodies is given by Hooke's law relating the deformation (finite strain) of a flexible body linearly with the force density (stress) acting upon it using the algebraic form of tensors [1, 28].

This paper intends to contribute to an integral view of the 'displacement domain of rigid bodies' and the 'deformation domain of flexible bodies'. In pursuit of this general goal, we consider the geometry of linear spatial deformations of flexible bodies in analogy with the geometry of linear spatial displacements of rigid bodies. Elastic deformations of a physical object depend on material attributes, as body shape and stiffness properties of the matter, as well as on the force distribution, contact geometry, and friction characteristics. Thus, the precise modeling and control of compliant mechanisms is a complex task [10, 34].

Within this paper, we only consider the geometry of elementary deformations: the material is assumed to be incompressible, orientation-preserving, linear and isotropic. In this case, the model of spatial deformations simplifies and the geometry of finite deformations is characterized by two independent components, termed *'(radial pure) shear'*, and *'(axial volumetric) squeeze'* which are comparable to the rotation-translation dichotomy of rigid body displacements. The paper builds on the distinction of the three binary number systems [30] of complex numbers, dual numbers, and double numbers to describe both rigid body as well as compliant body motions (Section 2). As complex numbers are capable to describe planar rotations, double numbers are capable to describe planar pure shears. Screw theory is rooted in principle of transference [4] where dual numbers are used together with complex number to describe spatial rigid body motion. Following a similar approach, a dual-double number formalism for modeling linear finite deformations of flexible bodies is proposed (Section 3). Consecutively, a sibling version of the Mozzi–Chasles theorem about the existence of an invariant axis for linear displacements is reported (Section 4) for the case of linear deformations together with variants of the 'cylindric' Euler–Rodrigues formula for spatial displacements [20, 5].

2 Three Number Systems

Three systems of *'binary numbers'* [18] have been indicated by Study [30] that feature a commutative multiplication.[1] They are denoted in unified form [18] as a p-number system $\mathbb{C}_\mathsf{p}$, with $\mathsf{p} \in \{-1, 0, +1\}$ and $\mathsf{q} := \sqrt{\mathsf{p}}$, such that $x + \mathsf{q} \cdot y \in \mathbb{C}_\mathsf{p}$ has the appearance of

$$x + \mathsf{q} \cdot y = \begin{cases} x + \mathsf{i} \cdot y \in \mathbb{C} & \text{for } \mathsf{p} = \mathsf{q}^2 = \mathsf{i}^2 = -1 \\ x + \varepsilon \cdot y \in \mathbb{D} & \text{for } \mathsf{p} = \mathsf{q}^2 = \varepsilon^2 = 0 \\ x + \mathsf{b} \cdot y \in \mathbb{B} & \text{for } \mathsf{p} = \mathsf{q}^2 = \mathsf{b}^2 = +1 \end{cases} . \tag{1}$$

For the systems of complex numbers $\mathbb{C}$, dual numbers $\mathbb{D}$, and double[2] numbers $\mathbb{B}$, and with real numbers $\phi, \mathring{x}, \varphi \in \mathbb{R}$, the exponentials of the general definition $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ take the form

$$\exp(\mathsf{i} \cdot \phi) = \cos(\phi) + \mathsf{i} \cdot \sin(\phi) \tag{2}$$

$$\exp(\varepsilon \cdot \mathring{x}) = 1 + \varepsilon \cdot \mathring{x} \tag{3}$$

$$\exp(\mathsf{b} \cdot \varphi) = \cosh(\varphi) + \mathsf{b} \cdot \sinh(\varphi) , \tag{4}$$

using properties of the powers $\mathsf{i}^k \in \{1, \mathsf{i}, -1, -\mathsf{i}\}$ and $\varepsilon^k \in \{1, \varepsilon, 0\}$ and $\mathsf{b}^k \in \{1, \mathsf{b}\}$ for the units i, ε, b and exponents $k \in \mathbb{N}_0$.[3,4] For a dual argument $\tilde{x} = x + \varepsilon \cdot \mathring{x} \in \mathbb{D} = \mathbb{R} + \varepsilon\mathbb{R}$, the dual extension of a function f fulfills the property [25]

$$f(\tilde{x}) = f(x) + \varepsilon \cdot \mathring{x} \cdot \tfrac{\partial}{\partial x}(f)(x) . \tag{5}$$

The form applies to (3) by $\exp(\varepsilon \cdot \mathring{x}) = \exp(0 + \varepsilon \cdot \mathring{x})$ so that $\exp(\varepsilon \cdot \mathring{x}) = \exp(0) + \varepsilon \cdot \mathring{x}$ holds.[5]

2.1 Complex Numbers and Rotations

The exponential for complex numbers of (2) is wrapped in the form of (3×3)-matrices, replacing the imaginary unit i by the generator matrix $\mathtt{i} := \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, as

$$\boldsymbol{R}_z = \exp(\phi \cdot \mathtt{i}) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ +\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathsf{SO}(3) \tag{6}$$

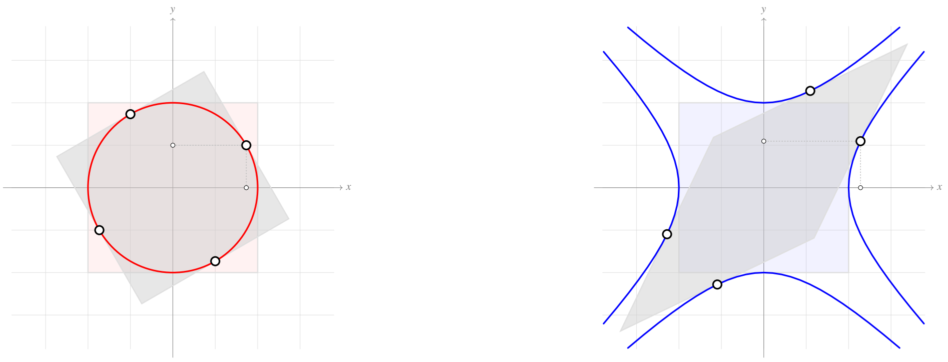
[1] These three systems are also called (members of the family of) *'generalized complex numbers'* [18].

[2] The concept of a double number $z = x + \mathsf{b} \cdot y$ with $\mathsf{b}^2 = +1$ is attributed to William K. Clifford [12]. The name is not used exclusively in literature, for example, the terms hyperbolic, perplex, split-complex, and pseudo-Euclidean are also used. The plane of double numbers is also called *Minkowski plane* [29].

[3] Since Equation 2 is well-known as *'Euler's formula'*, Equation 4 is also called *'hyperbolic Euler's formula'* [11].

[4] Equation 4 is extended to cover the four branches of the unit circle (Figure 1b) of the Minkowski plane in [11].

[5] By means of the 'Galilean' trigonometric functions $\operatorname{cosg}(x) := 1$ and $\operatorname{sing}(x) := x$, the dual exponential is expressed as $\exp(\varepsilon \cdot \mathring{x}) = \operatorname{cosg}(\mathring{x}) + \varepsilon \cdot \operatorname{sing}(\mathring{x})$ in consistent form [33].



(a) Complex circle and rotation for angle $\phi = \pi/6$. (b) Double circle and squeeze for charge $\varphi = \pi/6$.

Figure 1: Illustrations for the complex 'ordinary' unit circle and the double 'hyperbolic' unit circle and their kinematic interpretations as planar rotation (left) and planar deformation (right).

indicating a rotation about the axis $\hat{\boldsymbol{e}}_z = (0,0,1)^\top$. An illustration of the geometric effect of a z-rotation in the xy-plane is given in Figure 1a. In order to render rotations in space, for a rotation about an arbitrary axis $\hat{\boldsymbol{n}}$ with norm one $\hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} \in \mathbb{S}^2 \subset \mathbb{R}^3$, the matrix exponential is obtained from (6) replacing $\mathtt{i} \equiv \hat{\boldsymbol{e}}_z^\times$ by the generator

$$\hat{\boldsymbol{n}}^\times := \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \in \mathsf{so}(3) := \left\{ \boldsymbol{M} \in \mathbb{R}^{3\times 3} \mid \boldsymbol{M}^\top = -\boldsymbol{M}, \right\} . \tag{7}$$

A rotation matrix matrix $\boldsymbol{R} \in \mathsf{SO}(3)$ is given by Rodrigues' rotation formula in trigonometric form [5], with $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}^\top$ as the dyadic product, and by the similarity to $\boldsymbol{R}_z$ in the formulae:

$$\begin{aligned} \boldsymbol{R} = \exp(\phi \cdot \hat{\boldsymbol{n}}^\times) &= \cos(\phi) \cdot (-\hat{\boldsymbol{n}}^\times \cdot \hat{\boldsymbol{n}}^\times) + \sin(\phi) \cdot \hat{\boldsymbol{n}}^\times + (\hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}}) \\ &= \boldsymbol{Q} \cdot \exp(\phi \cdot \hat{\boldsymbol{e}}_z^\times) \cdot \boldsymbol{Q}^\top = \exp(\phi \cdot (\boldsymbol{Q} \cdot \hat{\boldsymbol{e}}_z)^\times) . \end{aligned} \tag{8}$$

The set of all rotation matrices forms the special-orthogonal group

$$\mathsf{SO}(3) := \left\{ \boldsymbol{B} \in \mathbb{R}^{3\times 3} \mid \boldsymbol{B}^\top \boldsymbol{B} = \boldsymbol{B}\boldsymbol{B}^\top = \boldsymbol{I},\ \det(\boldsymbol{B}) = +1 \right\} , \tag{9}$$

each preserves volume, due to $|\det(\boldsymbol{B})| = 1$, and orientation, due to the $\det(\boldsymbol{B}) > 0$ property.

2.2 Double Numbers and Squeezes

The exponential for double numbers in (4) is wrapped in form of (3×3)-matrices, replacing the double unit $\mathtt{b}$ by the generator matrix $\mathtt{b} := \left(\begin{smallmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ (identical to $\boldsymbol{A}_1$ of Appendix C) as

$$\boldsymbol{E}_\pm = \exp\big(\varphi \cdot \mathtt{b}\big) = \begin{pmatrix} \cosh\varphi & +\sinh\varphi & 0 \\ +\sinh\varphi & \cosh\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathsf{SYM}_1(3) \tag{10}$$

indicating a *squeeze transform*[6] in the plane $\hat{\boldsymbol{e}}_z^\perp$. In Figure 1b, the geometric effect of a z-squeeze is illustrated. In contrast to the group of rotation matrices in Equation 9, the set of special matrices

$$\mathsf{SYM}_1(3) := \left\{ \boldsymbol{B} \in \mathbb{R}^{3\times 3} \mid \boldsymbol{B}^\top = \boldsymbol{B},\ \boldsymbol{B} \succ 0,\ \det(\boldsymbol{B}) = +1 \right\} , \tag{11}$$

of which each preserves volume and orientation, does not form a group.[7] In analogy to Rodrigues' rotation formula (8), the exponential of a traceless symmetric matrix of the set

$$\mathsf{sym}_0(3) := \left\{ \boldsymbol{M} \in \mathbb{R}^{3\times 3} \mid \boldsymbol{M}^\top = \boldsymbol{M},\ \mathrm{trace}(\boldsymbol{M}) = 0 \right\} \tag{12}$$

is given by the closed-form given via its eigendecomposition as restated in the next theorem.

[6]A *squeeze transform* is alternately called a *hyperbolic rotation* or a *pure shear*. A *simple shear* is a combination of pure shear and a rotation and does not feature a symmetric matrix.

[7]Symmetric matrices are not closed with respect to matrix multiplication, the product of two symmetric matrices $\boldsymbol{B}, \boldsymbol{C} \in \mathsf{SYM}(3)$ is again symmetric only if $\boldsymbol{B}\boldsymbol{C} = \boldsymbol{C}\boldsymbol{B}$. Example $\boldsymbol{B} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\boldsymbol{C} = \left(\begin{smallmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{smallmatrix}\right)$ with $\boldsymbol{B}\boldsymbol{C} \neq \boldsymbol{C}\boldsymbol{B}$.

Theorem 1 (Symmetric exponential)**.** *The exponential of a traceless symmetric* $\theta \cdot \boldsymbol{S} \in \mathsf{sym}_0(3)$ *is a special symmetric* $\boldsymbol{E} \in \mathsf{SYM}_1(3)$*, with* $\det(\boldsymbol{E}) = \det(\exp(\theta \boldsymbol{S})) = \exp(\mathrm{trace}(\theta \boldsymbol{S})) = 1$*, given by*

$$\boldsymbol{E} = \exp\big(\theta \cdot \boldsymbol{S}\big) = \exp(\theta \cdot \lambda_A) \cdot (\hat{\boldsymbol{q}}_A \otimes \hat{\boldsymbol{q}}_A) + \exp(\theta \cdot \lambda_B) \cdot (\hat{\boldsymbol{q}}_B \otimes \hat{\boldsymbol{q}}_B) + \exp(\theta \cdot \lambda_A) \cdot (\hat{\boldsymbol{q}}_C \otimes \hat{\boldsymbol{q}}_C) \,,$$

eigenvalues $\lambda_A, \lambda_B, \lambda_C \in \mathbb{R}$ *with* $\lambda_A + \lambda_B + \lambda_C = 0$*, and orthogonal eigenvectors* $\hat{\boldsymbol{q}}_A, \hat{\boldsymbol{q}}_B, \hat{\boldsymbol{q}}_C \in \mathbb{R}^3$*.*

The eigendecomposition of symmetric $\boldsymbol{S}$ reads $\boldsymbol{S} = \boldsymbol{Q} \cdot \mathrm{diag}(\boldsymbol{\lambda}) \cdot \boldsymbol{Q}^\top$, and the theorem follows via

$$\exp\big(\theta \cdot \boldsymbol{S}\big) = \exp\big(\theta \cdot (\boldsymbol{Q}\,\mathrm{diag}(\boldsymbol{\lambda})\,\boldsymbol{Q}^\top)\big) = \boldsymbol{Q} \cdot \exp(\theta \cdot \mathrm{diag}(\boldsymbol{\lambda})) \cdot \boldsymbol{Q}^\top = \sum_{\mathsf{X} \in \{\mathsf{A},\mathsf{B},\mathsf{C}\}} \exp(\theta \cdot \lambda_\mathsf{X}) \cdot (\hat{\boldsymbol{q}}_\mathsf{X} \otimes \hat{\boldsymbol{q}}_\mathsf{X}) \,,$$

with $\boldsymbol{Q} = (\hat{\boldsymbol{q}}_\mathsf{A} | \hat{\boldsymbol{q}}_\mathsf{B} | \hat{\boldsymbol{q}}_\mathsf{C}) \in \mathsf{SO}(3)$, vector $\boldsymbol{\lambda} = (\lambda_\mathsf{A}, \lambda_\mathsf{B}, \lambda_\mathsf{C})^\top \in \mathbb{R}^3$, and $\mathrm{diag}(\boldsymbol{\lambda}) = \begin{pmatrix} \lambda_\mathsf{A} & 0 & 0 \\ 0 & \lambda_\mathsf{B} & 0 \\ 0 & 0 & \lambda_\mathsf{C} \end{pmatrix}$.

3 Principle of Transference

The transference principle for Euclidean geometry is recalled in Section 3.1 and varied for pseudo-Euclidean geometry in Section 3.2. The ordinary principle extends planar rotations (complex numbers) to spatial displacements (dual-complex numbers) and the converted principle extends planar squeezes (double numbers) to spatial deformations (dual-double numbers).

3.1 Ordinary Trigonometry and Displacements

The principle of transference connects the geometry of the sphere with the geometry of space: the dualification of a unit (direction) vector on the sphere $\hat{\boldsymbol{n}} \in \mathbb{S}^2 \subset \mathbb{R}^3$ to a dual (line) vector $\hat{\boldsymbol{\Lambda}} = \hat{\boldsymbol{n}} + \varepsilon \cdot \boldsymbol{m}$ is an element of the 'Study sphere' [24], $\hat{\boldsymbol{\Lambda}} \in \mathsf{T}(\mathbb{S}^2) \subset \mathbb{D}^3 = \mathbb{R}^3 + \varepsilon\mathbb{R}^3$. The principle is attributed to Kotel'nikov and Study and has been stated [25] with a focus on geometry as:

"All valid laws and formulae relating to a system of intersecting unit line vectors (hence involving real variables) are equally valid to an equivalent system of skew unit line vectors, if each real variable in the formulae is replaced by the corresponding dual variable"

A condensed statement [4] in algebraic terms is *"All identities of ordinary trigonometry hold true for dual angles"*. The concept of a dual angle is recalled next, using the notation $\tilde{x} \in \mathbb{D}$ from (5).

Definition 1 (Dual angle)**.** *A* 'dual angle' *is a dual number* $\tilde{\phi} = \phi + \varepsilon \cdot s \in \mathbb{D}$ *of a primal part* $\phi \in \mathbb{R}$ *indicating an angle and a dual part* $s \in \mathbb{R}$ *indicating a shift.*

For attributing a formula to this principle, the complex exponential of (2) is combined with the dual exponential of (3) to the dual-complex exponential function

$$\begin{aligned} \hat{\tilde{z}} = \exp(\mathrm{i} \cdot \tilde{\phi}) &= \exp(\mathrm{i} \cdot (\phi + \varepsilon \cdot s)) = \exp(\mathrm{i} \cdot \phi) + \varepsilon \cdot s \cdot \exp(\mathrm{i} \cdot \phi) \cdot \mathrm{i} \\ &= \big(\cos(\phi) + \mathrm{i} \cdot \sin(\phi)\big) + \varepsilon \cdot s \cdot \big(-\sin(\phi) + \mathrm{i} \cdot \cos(\phi)\big) \,, \end{aligned} \tag{13}$$

compliant with the property of dual-number functions in (5). The geometric image of the dual exponential function $\exp(\mathrm{i} \cdot \tilde{\phi})$ is illustrated as the dual-complex unit circle – identified as the tangent bundle of the ordinary unit circle $\mathsf{T}(\mathbb{S}^1)$ – in Figure 2a. Equation 13 can be considered as a 'prototype' for cylindric displacements along a fixed axis in space – the coordinate axis $\hat{\boldsymbol{e}}_z$ is chosen per convention [5]: For this spatial interpretation, the circle tangents are turned about $\pi/2$ 'out of the plane' so that a *'unit cylinder'* is achieved (Figure 4a).

Applying the transference principle to Rodrigues' rotation formula of Equation 8, the line transform in terms of (6×6)-adjoint matrices is obtained [5]. In particular, line transforms originate in the concept of *'motor calculus'* [31], for which a *motor* can be understood as the application of a dual angle along an oriented line in space $\hat{\boldsymbol{\Lambda}}$ as outlined in the next definition.

Definition 2 (Motor)**.** *A* 'motor' *is given by the dual product of (i) a 'dual angle'* $\tilde{\phi} = \phi + \varepsilon \cdot s \in \mathbb{D}$ *(a dual scalar) and (ii) an 'axis' (line)* $\hat{\boldsymbol{\Lambda}} = \hat{\boldsymbol{n}} + \varepsilon \cdot \boldsymbol{m} \in \mathbb{D}^3$ *(a dual vector). A motor maps one unit line onto another.*

(a) Complex circle with tangents. (b) Double circle with tangents.

Figure 2: Illustration of the tangent bundle of complex ('ordinary') unit circle and of double ('hyperbolic') unit circle. The example rotation and squeeze for $\pi/6$ match with Figure 1. The gray quadrilaterals in Fig. 2a and 2b are formed by (*'ordinary'* and *'hyperbolic'*) *orthogonal* vectors.

The computation of a motor's dual angle and dual vector from a displacement using the (6×6)-adjoint representation as a line transform is reported [6]. For point transforms, the usual representation as (4×4)-matrices for homogeneous point coordinates $\boldsymbol{p}=(x,y,z,1)^\top$ is obtained. A point transform $\boldsymbol{D}_z$ along the axis $\hat{\boldsymbol{e}}_z$ is determined via the matrix exponential, where its argument is given as a 'helical screw' displacement $\phi\cdot\hat{\$}_z^{(h)}$ with $\hat{\$}_z^{(h)}=\hat{\boldsymbol{e}}_z+\varepsilon\cdot h\cdot\hat{\boldsymbol{e}}_z$ and pitch $h:=s/\phi$, or as 'cylindric motor' displacement $\tilde{\phi}\cdot\hat{\boldsymbol{\Lambda}}_z$ with $\hat{\boldsymbol{\Lambda}}_z=\hat{\boldsymbol{e}}_z+\varepsilon\cdot\boldsymbol{0}$ in the formulae

$$\boldsymbol{D}_z=\exp\big(\mathrm{cross}(\phi\cdot\hat{\$}_z^{(h)})\big)=\exp\big(\phi\cdot\mathtt{i}_4+s\cdot\mathtt{t}_4\big)=\begin{pmatrix}\cos\phi & -\sin\phi & 0 & 0\\ +\sin\phi & \cos\phi & 0 & 0\\ 0 & 0 & 1 & s\\ 0 & 0 & 0 & 1\end{pmatrix}\in\mathsf{SE}(3)\,. \tag{14}$$

The term $\mathrm{cross}(\hat{\$}_z^{(h)})=\begin{pmatrix}0 & -1 & 0 & 0\\ +1 & 0 & 0 & 0\\ 0 & 0 & 0 & h\\ 0 & 0 & 0 & 0\end{pmatrix}$ is an instance of the matrix representation $\mathrm{cross}(\hat{\$}^{(h)})=\begin{pmatrix}\hat{\boldsymbol{n}}^\times & \boldsymbol{m}\\ \boldsymbol{0} & 0\end{pmatrix}$ of a unit screw $\hat{\$}^{(h)}=\hat{\boldsymbol{n}}+\varepsilon\cdot\boldsymbol{m}^{(h)}=\hat{\boldsymbol{n}}+\varepsilon\cdot(\boldsymbol{a}\times\hat{\boldsymbol{n}}+h\cdot\hat{\boldsymbol{n}})$, extending the skew-symmetric $\hat{\boldsymbol{n}}^\times$ of (7). The second argument uses the auxiliary matrices $\mathtt{i}_4:=\begin{pmatrix}0 & -1 & 0 & 0\\ +1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$ and $\mathtt{t}_4:=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$.

3.2 Double Trigonometry and Deformations

For characterizing the geometry of linear deformations, the 'flexible sibling' of the transference principle can be formulated in analogy to its 'rigid counterpart' as the conjecture:

> *"Valid formulae relating to linear deformations of a* planar *elastic body (involving double variables) are equally valid to linear deformations of a* spatial *elastic body, if each double variable in the formulae is replaced by the corresponding dual-double variable."*

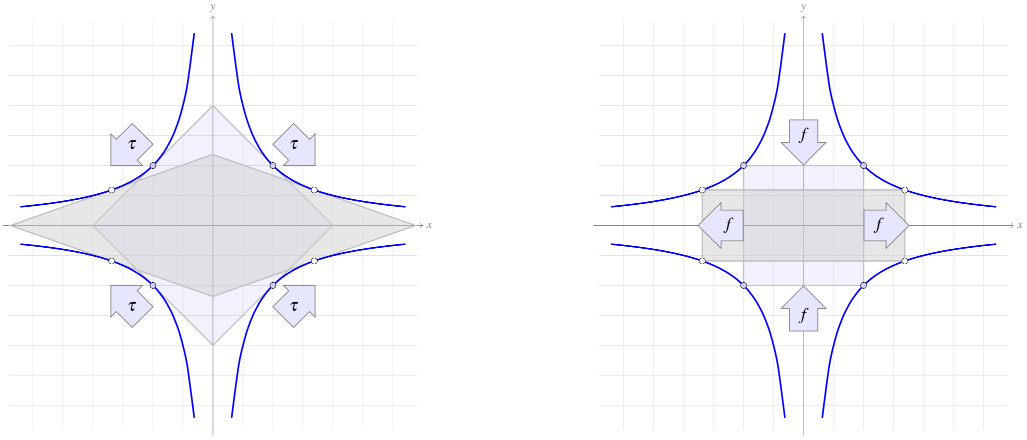
In analogy to the brief form in Section 3.1, the principle is converted into the conjecture *"All identities of double trigonometry hold true for dual charges"*. The concept of a dual charge given in next definition is the 'flexible version' corresponding to the dual angle in Definition 1.

Definition 3 (Dual charge)**.** *A* 'dual charge' *is a dual number $\tilde{\varphi}=\varphi+\varepsilon\cdot g\in\mathbb{D}$ of a primal part $\varphi\in\mathbb{R}$ indicating a radial charge and a dual part $g\in\mathbb{R}$ indicating an axial charge.*

For attributing a formula to this principle, the double exponential of Equation 4 is combined with the dual exponential of Equation 3 to the dual-double exponential function

$$\begin{aligned}\hat{\underset{\sim}{z}}=\exp(\mathsf{b}\cdot\tilde{\varphi})&=\exp(\mathsf{b}\cdot(\varphi+\varepsilon\cdot g))=\exp(\mathsf{b}\cdot\varphi)+\varepsilon\cdot g\cdot\exp(\mathsf{b}\cdot\varphi)\cdot\mathsf{b}\\&=\big(\cosh(\varphi)+\mathsf{b}\cdot\sinh(\varphi)\big)+\varepsilon\cdot g\cdot\big(\sinh(\varphi)+\mathsf{b}\cdot\cosh(\varphi)\big)\end{aligned} \tag{15}$$

with the property of dual-number functions in (5). Next to the image of (13) in Figure 2a, the image of $\exp(\mathsf{b}\cdot\tilde{\varphi})$ in (15) as the dual-double unit circle is illustrated as the tangent bundle of



(a) Forces τ tangential with object surfaces induce a deformation of pure shear type.

(b) Forces f normal to object surfaces induce a deformation of compression type.

Figure 3: Interpretation of a planar squeeze transform as a deformation of pure shear and of planar compression. The images are obtained from Fig. 1b by turning about $\pi/4$ and scaling about $\sqrt{2}$.

the four-branched unit hyperbola in Figure 2b. If a linear deformation $\boldsymbol{E}$ preserves volume and orientation of a body with $\det(\boldsymbol{E}) = 1$ and $\boldsymbol{E} \in \mathsf{SYM}_1(3)$, Theorem 1 tells by

$$\boldsymbol{E} = \exp\big(\theta \cdot \boldsymbol{S}\big) = \boldsymbol{Q} \cdot \exp\big(\theta \cdot \operatorname{diag}(\lambda_\mathsf{A}, \lambda_\mathsf{B}, \lambda_\mathsf{C})\big) \cdot \boldsymbol{Q}^\top$$

with $\lambda_\mathsf{A} + \lambda_\mathsf{B} + \lambda_\mathsf{C} = 0$, or in terms of $\boldsymbol{\lambda} = (\lambda_\mathsf{A}, \lambda_\mathsf{B}, \lambda_\mathsf{C})^\top$ with $\boldsymbol{\lambda} \perp \mathbf{1}$, that the spatial deformation $\boldsymbol{E}$ is similar to a deformation of the unit sphere into an axis-aligned, isovolumetric ellipsoid of principal axes of lengths λ_A, λ_B, and λ_C. Considering a deformation $\boldsymbol{E}_z$ with $\boldsymbol{Q} = \boldsymbol{I}$, and $\boldsymbol{E}_z = \exp\big(\theta \cdot \operatorname{diag}(\lambda_\mathsf{A}, \lambda_\mathsf{B}, \lambda_\mathsf{C})\big)$, the constraint $\operatorname{trace}(\operatorname{diag}(\boldsymbol{\lambda})) = 0$ is decomposed into a primal and a dual part. With $\boldsymbol{h}_z := \left(\begin{smallmatrix} +1 \\ -1 \\ 0 \end{smallmatrix}\right) = \mathbf{1} \times \hat{\boldsymbol{e}}_z$ and $\boldsymbol{k}_z = \left(\begin{smallmatrix} +1 \\ +1 \\ -2 \end{smallmatrix}\right) := \mathbf{1} \times \boldsymbol{h}_z$ and letting

$$\mathtt{h}_z := \operatorname{diag}\big(\boldsymbol{h}_z\big) = \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathtt{k}_z := \operatorname{diag}\big(\boldsymbol{k}_z\big) = \begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{16}$$

corresponding to the Gell-Mann matrices $\boldsymbol{A}_3$ and $\boldsymbol{A}_8$ in Appendix C, the *'radial shear'* deformation $\boldsymbol{H}_z$ in the plane $\hat{\boldsymbol{e}}_z^\perp$ and the *'axial compression'* deformation $\boldsymbol{K}_z$ along the axis $\hat{\boldsymbol{e}}_z$ read

$$\boldsymbol{H}_z = \exp\big(\varphi \cdot \mathtt{h}_z\big) = \begin{pmatrix} e^{\varphi} & 0 & 0 \\ 0 & e^{-\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{K}_z = \exp\big(g \cdot \mathtt{k}_z\big) = \begin{pmatrix} e^{g} & 0 & 0 \\ 0 & e^{g} & 0 \\ 0 & 0 & e^{-2g} \end{pmatrix}. \tag{17}$$

In terms of these two components, the deformation $\boldsymbol{E}_z$ is decomposed into the product

$$\boldsymbol{E}_z = \exp\big(\theta\boldsymbol{\lambda}\big) = \boldsymbol{H}_z \cdot \boldsymbol{K}_z = \exp\big(\varphi \cdot \mathtt{h}_z + g \cdot \mathtt{k}_z\big) = \begin{pmatrix} \exp(g+\varphi) & 0 & 0 \\ 0 & \exp(g-\varphi) & 0 \\ 0 & 0 & \exp(-2g) \end{pmatrix}, \tag{18}$$

with determinant $\det(\boldsymbol{E}_z) = e^{\varphi+g} \cdot e^{\varphi-g} \cdot e^{-2g} = 1$. Two realizations of the planar shear $\boldsymbol{H}_z$ of (17) are given by Figure 3: A deformation in the plane ($\pi/4$-rotated to Figure 1b and Figure 2b) is achieved by applying forces in parallel tangential with the surfaces in Figure 3a and normal to the surfaces in Figure 3b. Since the axial compression $\boldsymbol{K}_z$ in space of (17) distributes matter from one source dimension onto two sink dimensions, a graphical interpretation would be given with the surface of revolution obtained by using a (scaled) unit hyperbola as the generatrix along the central axis $\hat{\boldsymbol{e}}_z$ (other axes in Figure 3). The concept of a morphor in next definition is the 'flexible version' corresponding to a motor of Definition 2.

Definition 4 (Morphor)**.** *A* 'morphor' *is given by (the dual part of) the dual product of (i) a 'dual charge' $\tilde{\varphi} = \varphi + \varepsilon \cdot g \in \mathbb{D}$ (a dual scalar) and (ii) a 'parcel' $\hat{\boldsymbol{\Gamma}} = \boldsymbol{x} + \varepsilon \cdot \boldsymbol{y} \in \mathbb{D}^3$ (a dual vector). A morphor maps one unit cell onto another.*

(a) Dual-complex unit circle illustrated of as a *cylinder* in space indicating the set of all displacement sizes ('dual angles'). A cylinder along the axis of the rigid Mozzi–Chasles Theorem 2 is invariant with respect to screw displacements about this axis.

(b) Dual-double unit circle illustrated as a *maltese cross* in space indicating the set of all deformation sizes ('dual charges'). The shape is the basis for the flexible Mozzi–Chasles Theorem 3 for oriented-volume preserving deformations.

Figure 4: Spatial images of the dual extensions for the unit circle of the plane of complex and double numbers. The shapes are spatial interpretations of the planar illustrations in Figure 2 turned about $\pi/4$ along the z-axis. While the case of a pure rotation in Fig. 2a corresponds to the xy-plane at $z=0$, a pure shear in Fig. 2b and Fig. 3 corresponds to the xy-planes at $z=+1$ and $z=-1$.

By means of the dual vector $\boldsymbol{k}_z+\varepsilon\cdot\boldsymbol{h}_z$ associated to $\hat{\boldsymbol{e}}_z$, the exponential argument $\varphi\cdot\mathtt{h}_z+g\cdot\mathtt{k}_z$ in (18) is obtained as the dual part of the product

$$(\varphi+\varepsilon\cdot g)\cdot\operatorname{diag}\big(\boldsymbol{k}_z+\varepsilon\cdot\boldsymbol{h}_z\big)=\varphi\cdot\mathtt{k}_z+\varepsilon\cdot\big(\varphi\cdot\mathtt{h}_z+g\cdot\mathtt{k}_z\big)\ .$$

The parameters of the spatial deformation $\boldsymbol{E}_z=\boldsymbol{H}_z\cdot\boldsymbol{K}_z$ of (18), combined of an axial shear $\boldsymbol{H}_z$ and a spatial compression $\boldsymbol{K}_z$, are illustrated in Figure 4b: The figure is named 'maltese cross' here and can be considered as the product set of two orthgonal, axis aligned unit hyperbolas (Figure 3).

4 Reconfigurations in Space

This section reports theorems for the finite transforms for rigid bodies and for flexible bodies. In Appendix A, the theorems for the instantaneous domain are stated for sake of completion.

4.1 Rigid Body Displacements

Theorem 2 (Serial spatial displacements)**.** *Any set of displacements applied in series equals a combination of rotation and translation about a unique axis (Mozzi–Chasles). The combined displacement can be rendered as a motor along that axis (Euler–Rodrigues) using the formulae:*

$$\begin{aligned}\boldsymbol{D}=\exp\big(\operatorname{cross}(\phi\cdot\hat{\$})\big)=\exp\big(\operatorname{cross}(\tilde{\phi}\cdot\hat{\boldsymbol{\Lambda}})\big)&=\boldsymbol{P}_4\cdot\boldsymbol{D}_z\cdot\boldsymbol{P}_4^{-1}\\&=\boldsymbol{P}_4\cdot\exp\big(\phi\cdot\mathtt{i}_4\big)\cdot\boldsymbol{P}_4^{-1}\cdot\boldsymbol{P}_4\cdot\exp\big(s\cdot\mathtt{t}_4\big)\cdot\boldsymbol{P}_4^{-1}=\boldsymbol{T}_4\cdot\boldsymbol{R}_4\ .\end{aligned}\tag{19}$$

The second term yields the 'screw-like' and the third term the 'cylinder-like interpretation' [20] interpretation of a displacement. The fourth term indicates the similarity to $\boldsymbol{D}_z$ of Equation 14 via $\boldsymbol{P}_4=\begin{pmatrix}\boldsymbol{Q}&\boldsymbol{p}\\\boldsymbol{0}&1\end{pmatrix}$. The fifth term is the UQ decomposition with respect to $\boldsymbol{T}_4=\begin{pmatrix}\boldsymbol{I}&\boldsymbol{t}\\\boldsymbol{0}&1\end{pmatrix}$ and $\boldsymbol{R}_4=\begin{pmatrix}\boldsymbol{R}&\boldsymbol{0}\\\boldsymbol{0}&1\end{pmatrix}$.

4.2 Flexible Body Deformations

For brevity, the abbreviations $\boldsymbol{n}^{\circledcirc}:=-(\boldsymbol{n}^{\times})^2$ and $\boldsymbol{n}^{\oslash}:=\boldsymbol{n}\otimes\boldsymbol{n}$ are used [5]. For an orthonormal basis $(\hat{\boldsymbol{q}}_{\mathsf{A}}|\hat{\boldsymbol{q}}_{\mathsf{B}}|\hat{\boldsymbol{q}}_{\mathsf{C}})\in\mathsf{SO}(3)$, the identities $\hat{\boldsymbol{q}}_{\mathsf{C}}^{\oslash}=\hat{\boldsymbol{q}}_{\mathsf{B}}^{\circledcirc}-\hat{\boldsymbol{q}}_{\mathsf{A}}^{\oslash}=\hat{\boldsymbol{q}}_{\mathsf{A}}^{\circledcirc}-\hat{\boldsymbol{q}}_{\mathsf{B}}^{\oslash}$ and $\hat{\boldsymbol{q}}_{\mathsf{C}}^{\circledcirc}=\hat{\boldsymbol{q}}_{\mathsf{A}}^{\oslash}+\hat{\boldsymbol{q}}_{\mathsf{B}}^{\oslash}$ follow immediately with $\hat{\boldsymbol{q}}_{\mathsf{A}}^{\oslash}+\hat{\boldsymbol{q}}_{\mathsf{B}}^{\oslash}+\hat{\boldsymbol{q}}_{\mathsf{C}}^{\oslash}=\boldsymbol{I}$ and $\boldsymbol{I}=\hat{\boldsymbol{q}}_{\mathsf{C}}^{\circledcirc}+\hat{\boldsymbol{q}}_{\mathsf{C}}^{\oslash}$. Similar to the cross matrix identity $(\boldsymbol{Q}\cdot\boldsymbol{n})^{\times}=\boldsymbol{Q}\cdot(\boldsymbol{n}^{\times})\cdot\boldsymbol{Q}^{\top}$, the matrices $\boldsymbol{n}^{\circledcirc}$ and $\boldsymbol{n}^{\oslash}$ are characterized by the equations

$$(\boldsymbol{Q}\cdot\boldsymbol{n})^{\circledcirc}=\boldsymbol{Q}\cdot(\boldsymbol{n}^{\circledcirc})\cdot\boldsymbol{Q}^{\top}\qquad\qquad(\boldsymbol{Q}\cdot\boldsymbol{n})^{\oslash}=\boldsymbol{Q}\cdot(\boldsymbol{n}^{\oslash})\cdot\boldsymbol{Q}^{\top}\ .\tag{20}$$

	Rigid body kinematics	*Flexible body kinematics*
Finite transform	displacement $\boldsymbol{D} \cong (\boldsymbol{R}, \boldsymbol{t})$	deformation $\boldsymbol{E} \cong (\boldsymbol{H}, \boldsymbol{K})$
Primary operand	line (direction + moment)	point ('no primal direction')
Radial transform Axial transform	rotation (around axis) $\boldsymbol{R}$ translation (along axis) $\boldsymbol{t}$	pure shear (tangential) $\boldsymbol{H}$ compression (normal) $\boldsymbol{K}$
Special set	unit sphere with $x^2+y^2+z^2=1$	'unit plane' with $x+y+z=1$
Compound variable	dual angle $\tilde{\phi} = \phi + \varepsilon \cdot s$ radial $\phi = \int \omega$ and axial $s = \int v$	'dual charge' $\tilde{\varphi} = \varphi + \varepsilon \cdot g$ radial $\varphi \cong \tau$ and axial $g \cong f$
Geometry / numbers	ordinary / complex	hyperbolic / double
Exp. argument	skew-symmetric matrix (motor)	symmetric-traceless matrix ('morphor')
Eigenspaces	affine line in space (spear)	three orthogonal axes ('parcel')
Transference	attach tangent bundle to unit circle	planar double circle to maltese cross
Transform types	rotation to cylindric motion	planar squeeze to spatial deformation

Table 1: Tabular comparison of displacements and deformations.

We consider the symmetric-traceless matrices $\mathtt{h}_z = \operatorname{diag}(\boldsymbol{h}_z)$ and $\mathtt{k}_z = \operatorname{diag}(\boldsymbol{k}_z)$ of (16) from a more general point of view by means of the matrix function 'quad' defined as

$$\operatorname{quad}(\boldsymbol{z}, \boldsymbol{y}) := \boldsymbol{z}^{\circledcirc} - 2 \cdot \boldsymbol{y}^{\oslash} \, . \tag{21}$$

Here, $\boldsymbol{z}$ and $\boldsymbol{y}$ indicate the compressive radial and the compressive axial directions. The direction $\boldsymbol{x}$ follows by $\boldsymbol{x} = \boldsymbol{y} \times \boldsymbol{z}$. For the case $\boldsymbol{z} \perp \boldsymbol{y}$, 'quad' has the particular form $\operatorname{quad}(\boldsymbol{z}, \boldsymbol{y}) = (\boldsymbol{y} \times \boldsymbol{z})^{\oslash} - \boldsymbol{y}^{\oslash}$, for the case $\boldsymbol{z} = \boldsymbol{y}$, the form is $\operatorname{quad}(\boldsymbol{z}, \boldsymbol{z}) = \boldsymbol{I} - 3 \cdot \boldsymbol{z}^{\oslash}$, both obtained via $\hat{\boldsymbol{q}}_{\mathsf{C}}^{\circledcirc} = \hat{\boldsymbol{q}}_{\mathsf{A}}^{\oslash} + \hat{\boldsymbol{q}}_{\mathsf{B}}^{\oslash}$. The diagonal generator matrices $\mathtt{h}_z = \operatorname{diag}(\boldsymbol{h}_z)$ and $\mathtt{k}_z = \operatorname{diag}(\boldsymbol{k}_z)$ are recovered by

$$\begin{aligned}
\mathtt{h}_z = \operatorname{diag}(\boldsymbol{h}_z) = \mathtt{H}_{xy} &:= \operatorname{quad}(\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_y) = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&0 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix} = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix} \\
\mathtt{k}_z = \operatorname{diag}(\boldsymbol{k}_z) = \mathtt{K}_{xy} &:= \operatorname{quad}(\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_y) = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&0 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{pmatrix} \, .
\end{aligned}$$

By means of the extended notions $\mathtt{h}_z \to \mathtt{H}_{xy}$ and $\mathtt{k}_z \to \mathtt{K}_{xy}$, the plane $\hat{\boldsymbol{e}}_z^{\perp}$ is specified via the bivector $\hat{\boldsymbol{e}}_x \wedge \hat{\boldsymbol{e}}_y$ including the orientation of radial compression and tension.[8] Using the identities in (20), the rule $\operatorname{quad}(\boldsymbol{Q} \cdot \boldsymbol{n}, \boldsymbol{Q} \cdot \boldsymbol{m}) = \boldsymbol{Q} \cdot \operatorname{quad}(\boldsymbol{n}, \boldsymbol{m}) \cdot \boldsymbol{Q}^{\top}$ is obtained which permits to state

$$\mathtt{H}_{Qxy} := \operatorname{quad}\big(\boldsymbol{Q} \cdot (\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_y), \boldsymbol{Q} \cdot \hat{\boldsymbol{e}}_y\big) \qquad \mathtt{K}_{Qxy} := \operatorname{quad}\big(\boldsymbol{Q} \cdot (\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_y), \boldsymbol{Q} \cdot (\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_y)\big) \, .$$

The concept 'parcel' $\hat{\boldsymbol{\Gamma}}$ in Definition 4 is proposed as $\hat{\boldsymbol{\Gamma}} := \boldsymbol{x} + \varepsilon \cdot \boldsymbol{y}$, identified with $\boldsymbol{x} \wedge \boldsymbol{y}$ for $\boldsymbol{x} = \boldsymbol{Q} \cdot \hat{\boldsymbol{e}}_x$ and $\boldsymbol{y} = \boldsymbol{Q} \cdot \hat{\boldsymbol{e}}_y$ that determines the unit cell of a deformation $\boldsymbol{E}$. The matrix form of $\hat{\boldsymbol{\Gamma}}$ is thus proposed (comparable to the skew-symmetric matrix form of a line) as the symmetric dual matrix $\operatorname{zq}(\boldsymbol{x} + \varepsilon \cdot \boldsymbol{y}) = \mathtt{K}_{Qxy} + \varepsilon \cdot \mathtt{H}_{Qxy} = \operatorname{zqk}(\boldsymbol{x}, \boldsymbol{y}) + \varepsilon \cdot \operatorname{zqh}(\boldsymbol{x}, \boldsymbol{y}) := \operatorname{quad}(\boldsymbol{x} \times \boldsymbol{y}, \boldsymbol{y}) + \varepsilon \cdot \operatorname{quad}(\boldsymbol{x} \times \boldsymbol{y}, \boldsymbol{x} \times \boldsymbol{y})$. With these preparations, the variation of Theorem 2 for elementary deformations is stated.

Theorem 3 (Parallel spatial deformations)**.** *Any set of deformations applied in parallel equals a combination of shear and compression about a parcel (Mozzi–Chasles). The combined deformation can be rendered as a morphor along such parcel (Euler–Rodrigues) using the formulae*

$$\begin{aligned}
\boldsymbol{E} = \boldsymbol{Q} \cdot \exp\big(\operatorname{diag}(\theta \cdot \boldsymbol{\lambda})\big) \cdot \boldsymbol{Q}^{\top} &= \boldsymbol{Q} \cdot \exp\big(\varphi \cdot \operatorname{diag}(\boldsymbol{h}_z) + g \cdot \operatorname{diag}(\boldsymbol{k}_z)\big) \cdot \boldsymbol{Q}^{\top} \\
&= \boldsymbol{Q} \cdot \exp\big(\varphi \cdot \mathtt{H}_{xy}\big) \cdot \boldsymbol{Q}^{\top} \cdot \boldsymbol{Q} \cdot \exp\big(g \cdot \mathtt{K}_{xy}\big) \cdot \boldsymbol{Q}^{\top} \\
&= \exp\big(\varphi \cdot \mathtt{H}_{Qxy}\big) \cdot \exp\big(g \cdot \mathtt{K}_{Qxy}\big) = \boldsymbol{H} \cdot \boldsymbol{K} \, .
\end{aligned} \tag{22}$$

The theorem follows from Theorem 1 with the orthogonal decomposition $\theta \cdot \boldsymbol{\lambda} = \varphi \cdot \boldsymbol{h}_z + g \cdot \boldsymbol{k}_z$ that exists for coplanar vectors $\boldsymbol{\lambda}, \boldsymbol{h}_z, \boldsymbol{k}_z \in \mathbf{1}^{\perp}$ with $\boldsymbol{h}_z \perp \boldsymbol{k}_z$.

[8] See the matrices $\boldsymbol{E}_{\pm}$ in Equation 10 and $\boldsymbol{H}_z$ in Equation 17 and their visualizations in Figures 1b and 3 as distinct examples of deformations in the same plane $\hat{\boldsymbol{e}}_z^{\perp}$ but with rotated eigenspaces.

4.3 Discussion

For linking Theorem 2 and Theorem 3 with the instantaneous theorems in Appendices A and B, an invariant motion state (constant object velocity and material flux) is assumed. In this case, Theorem 2 permits to integrate over a uniform displacement and Theorem 3 over a uniform deformation. In contrast to the non-symmetric matrix coefficients of displacements in Theorem 2, the symmetric matrix coefficients of deformations in Theorem 3 are not linear but quadratic with respect to the constituting vectors. In Table 1, the proposed analogy between finite displacements and elementary deformations is summarized.

5 Conclusions

The three systems of commutative binary numbers have been taken as a starting point to study the geometry of elementary deformations of elastic bodies in analogy to the geometry of linear displacements of rigid bodies. The principle of transference, which appears in screw theory to parametrize displacements in Euclidean space, has been converted into a novel conjecture suitable to parametrize deformations with pseudo-Euclidean characteristics in a similar manner. For this purpose, several sibling terms have been proposed. Pendants of the theorems by Mozzi–Chasles and Euler–Rodrigues have been stated for flexible finite kinematics based on these concepts. For the instantaneous domain, the duality of the spatial velocities (twists) and spatial force (wrenches) in connection with serial and parallel mechanisms [32, 13] is established. With this paper, a step towards a similar analogy for the finite domain of displacements and deformations has been proposed. The restriction to elementary deformations – assuming incompressible, orientation-preserving, linear, isotropic material and neglecting boundary conditions – permits to identify shear and compression as two independent components comparable to the rotation-translation dichotomy of displacements. With this structural observation, the conducted analogy consideration might contribute to ease the treatment of reconfigurations of compliant systems in the future.

A Instantaneous Domain

The term *'forque'* is used as a placeholder for a *'force'* or a *'torque'* [14].

Theorem 4 (Instantaneous Mozzi–Chasles)**.** *Any set of velocities applied in series equals a twist about a unique axis.*

Theorem 5 (Instantaneous Poinsot)**.** *Any set of forques applied in parallel equals a wrench about a unique axis.*

B Hooke Law

For linear 'one-dimensional' springs, the deformation x is given by Hooke's motion law $x = \frac{1}{k} \cdot f$ in terms of the spring's flexibility $\frac{1}{k}$ and f, the applied force. For linear-elastic, isotropic, three-dimensional bodies, the deformation $\boldsymbol{E}$ (finite strain $\boldsymbol{\varepsilon}$) is given by the law of Hooke, expressed via the linear relation

$$\boldsymbol{E} = \boldsymbol{\varepsilon} = \frac{1+\nu}{E} \cdot \boldsymbol{\sigma} - \frac{\nu}{E} \cdot \mathrm{trace}(\boldsymbol{\sigma}) \cdot \boldsymbol{I}\,, \tag{23}$$

where $\boldsymbol{\sigma}$ indicates the applied force density (stress), ν the Poisson ratio, and E the Young modulus.

C Gell-Mann Matrices

Gell-Mann [15] has defined eight traceless (3×3)-matrices connected with the (2×2)-matrices of Pauli. Among these eight, the three real symmetric matrices $\boldsymbol{A}_1$, $\boldsymbol{A}_3$, and $\boldsymbol{A}_8$

$$\boldsymbol{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{A}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{A}_8 = \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

are associated to the coordinate axis $\hat{\boldsymbol{e}}_z = (0,0,1)^\top$ and to the plane $\hat{\boldsymbol{e}}_z^\perp$.

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