A survey of empirical friction models for lubricated slotted joints in multibody dynamics simulations

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ABSTRACT

In recent times, many industrial applications demand innovative energy-efficient solutions. One of the main causes of energy loss is due to friction between body surfaces in contact. There is a great amount of research aimed to understand the friction mechanisms to allow its reliable prediction during simulation. In the Fifties and Sixties of the 20th Century, many investigations carried out experimental activities that led to coefficient of friction formulas for lubricated surfaces under a combination of sliding and rolling relative motion. The formulas have been mainly deduced by mathematical fitting of results obtained with experimental measurements on rolling disks and different load, lubricating and kinematic conditions. The purpose of this paper is twofold: on the one hand, reviewing semi-empirical formulas for the computation of friction coefficient in lubricated contact under different operating conditions; on the other hand, embodying and comparing, in multibody dynamics simulation environment, contact force models coupled with the metal-metal lubricated empirical friction formulas. The implementation of empirical formulas is straightforward and computationally efficient, but one can evaluate the performance of these models in characterising the dynamics of the lubricated joint. With this purpose, a multibody simulation of a scotch-yoke mechanism with non-ideal prismatic joint is conducted. The existence of clearance causes the dynamic behaviour of the system to be different from the ideal joint. The difference between each friction model is emphasized by the means of both simulation output and computation time.

Keywords: Contact dynamics, Lubricated joints, Joint clearance, Multibody dynamics, Friction models.

1 INTRODUCTION

During the 1950s and 1960s of the 20th Century, several experimental studies suggested many formulations of the friction coefficient for lubricated surfaces under a mixed relative motion of sliding and rolling. The formulas have been mainly deduced by mathematical fitting of results obtained with experimental measurements on rolling disks and different load, lubricating and kinematic conditions. The idea of a disk-based experimental apparatus for investigating the friction between surfaces in lubricated line contact dates back to Merritt [1]. His testing analogy between rolling disks and meshing gears inspired many investigators that improved and adapted to their purposes the original experimental apparatus based on disks test machines. For instance, Tabor *et al.* [2, 3, 4], Hersey [5, 6] offer an extensive historical review about experimental activities on rolling friction. Martin [7] discusses various analyses about the effects of rolling and sliding friction in gear drives. Many industrial applications demand innovative, energy-efficient solutions. Friction between bodies in contact with one other's surfaces is one of the major sources of energy loss. Thoughtful investigations have been dedicated to better understand the mechanisms of friction in order to comprehend these phenomena and provide accurate estimations. The emphasis of this study is on metal-metal lubricated joints, where a fluid film is interposed between contacting surfaces to reduce friction. Some researchers (e.g. Michlin [8], Valentini *et al.* [9], Flores *et al.* [10, 11], Zhao *et al.* [12, 13]) investigated the multibody dynamics modeling of lubricated joints. Within multiody dynamics simulations, empirically based formulas offer a concise and computationally efficient engineering approach to simulate complex friction phenomena in metal-metal lubricated slotted joints. These formulas have often been adopted in gear dynamics simulation (e.g. [14, 15]). However, a survey on their application in multibody dynamics simulation does not seem to be available. With the focus on the multibody dynamics simulation of a scotch-yoke linkage with a straight slot, the purpose of this paper is twofold:

- to review semi empirical formulas and provide a compiled list of friction coefficient formulations deduced mainly by fitting of experimental data;
- to embody and compare, in multibody dynamics simulation environment, contact force models (e.g. [16, 17]) coupled with the metal-metal lubricated empirical friction formulas and evaluate the possibility of employing these formulations bypassing the solution of the elastohydrodynamic problem (e.g. Reynolds equation).

2 CONTACT FORCE MODELS

The elastic contact force models play an important role in multibody dynamics simulation of mechanical systems with clearances in the kinematic pairs. Many investigations and reviews have been dedicated to the topic (e.g. [16, 17, 18, 19]) and it is out of the scope of this work to analyse the plethora of formulation available in literature. One of the most extensively used formulation for contact force is the one proposed by Lankarani and Nikravesh [20]. The kinetic energy lost as a result of internal damping allows to estimate the hysteresis-damping factor. The loss of kinetic energy is evaluated based on the kinetic energy prior to and following the contact (i.e restitution coefficient c_r) as well as the normal component of the relative contact velocity at the beginning of the contact-impact event $\dot{\delta}$. In particular, the normal force in cylindrical contact is evaluated as

$$F_n = K\delta^n \left(1 + \frac{3}{4}\frac{\dot{\delta}}{\dot{\delta}_0}(1 - c_r^2)\right) \tag{1}$$

where K is the contact-stiffness parameter, δ represent the relative indentation between bodies in contact, and n is the exponent representing the non-linear relation between penetration and force depending on the material and the local geometrical properties of contacting bodies. The contact stiffness parameter K can be either evaluated experimentally or analytically and for the contact between two cylinder requires an iterative solution of an implicit equation. In order to overcome this shortcomings, Autiero *et al.* [17] expressed the force indentation relations with parametric fitted polynomial of the type

$$F_n = K\delta^n \tag{2}$$

in order to speed up the computation with relatively low error. A dry frictionless contact produces only a force that is normal to the surfaces in contact; conversely, friction simultaneously generates a tangential component. Contact force models for cylinders are employed in the evaluation of normal component, while for the friction force some precautions are needed to avoid numerical difficulties. Among the plethora of possibility for the evaluation of friction force preventing numerical problems in dry [21] and lubricated [22] contacts, only the one chosen for the application in Section 5 is briefly discussed. The model, reported in [11], consists in a modified Coulomb force expressed as

$$F_T = -c_d \mu F_n \frac{\vec{V}_s}{|\vec{V}_s|} \tag{3}$$

where μ represents the friction coefficient and the dynamic correction factor c_d prevents the friction force from changing direction for negligible tangential velocity values, minimizing numerical issues. This factor is defined as

$$c_d = \begin{cases} 0 & \text{if} \quad |\vec{V}_s| \leq v_0 \\ \frac{|\vec{V}_s| - V_0}{V_1 - V_0} & \text{if} \quad V_0 \leq |\vec{V}_s| \leq V_1 \\ 1 & \text{if} \quad |\vec{V}_s| \leq V_0 \end{cases} \tag{4}$$

in which V_0 and V_1 are the threshold velocities for the sliding speed. It is important to stress that the Equation 3 does not take into account stiction, but only sliding.

3 REVIEW OF FRICTION COEFFICIENT FORMULATIONS

Many researchers conducted experimental work in the fifties and sixties of the 20th century that led to coefficient of friction formulas for lubricating surfaces under a combination of relative sliding and rolling motion[4]. The formulas were mainly derived by mathematical fitting of results obtained from experimental measurements of rolling disks under different loads, lubricants and kinematic conditions. Most sliding contacts are lubricated, in this case friction decreases with increasing sliding speed up to the formation of a sufficiently thick lubricant film. After that, a mixed or full film condition is observed and the friction coefficient can either be constant, raise or decrease with increased sliding speed [22]. This phenomenon was described by Stribeck [23], whose name is often associated with sliding friction in lubricated contacts running under boundary, mixed, and full film conditions. The lubrication regime and its effect on friction is depicted in Figure 1, where the so-called Stribeck curve is qualitatively illustrated. In boundary lubrication (region A), the load is mostly carried by asperities and the lubricant act at monomolecular scale [24]. Hence, the type of the lubricant and the mating surfaces are the primary factor of boundary friction, which is relatively unaffected by speed, load, and lubricant viscosity. At low loads and relatively high speed, hydrodynamic lubrication regime is reached (region C). A thick layer of lubricant now completely separated the surfaces and the friction is calculated from Reynolds equations. The region B is called mixed lubrication where the load is shared among fluid and asperities.

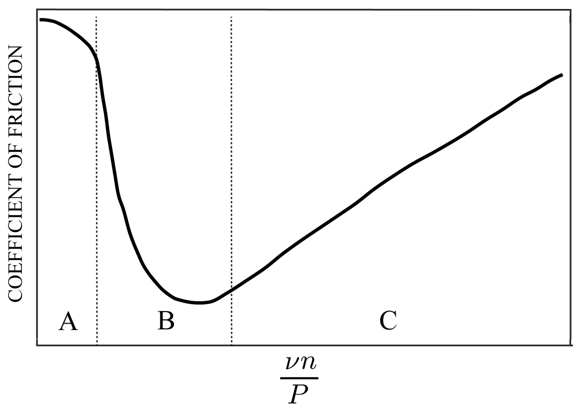

Figure 1. Stribeck curve: friction coefficient varying the ratio between the product viscosity times rotational speed (νn) and the load (P) .

Despite a few exceptions, mixed lubrication region is where the majority of gear applications are found. Low speed gears that are severely loaded exhibit symptoms of being in the boundary region, while lightly loaded high speed gears may achieve complete fluid film lubrication. All friction coefficient models, obtained for gear application, are evaluated for mixed lubrication conditions, which means that in quantitative terms the specific film thickness (i.e $\lambda = h_{min}/\sigma$, with h_{min} and σ the minimum film thickness and composite surface roughness respectively) assume values between 0.5 and 3 [25]. The authors of this paper are well aware that more sophisticated models have been developed for mixed lubrication condition and available in the current technical literature, especially regarding the definition of the friction coefficient for two mating gears [26, 27]. However, the present investigation focuses on simplified models that could be conveniently embedded in the algorithms for the dynamic simulations of multibody systems working under non stationary conditions, with the intent of establishing a trade-off between computational efficiency and accuracy. Moreover, the formulas herein reviewed have been also embodied in gear dynamics modelling and experimentally validated [7, 28, 14]. In table 1 the reviewed friction coefficient formulations are displayed reporting units and application bound where available. Referring to the table, μ represents the friction coefficient, ν_0 the kinematic viscosity at mean surface temperature, η_0 the dynamic viscosity at mean surface temperature, W the normal load, V_s the sliding speed, V_Σ the sum of profiles velocities, V_r the rolling speed (i.e $V_r = V_\Sigma/2$), S the surface roughness and R the effective radius of curvature (i.e. $R = R_i R_j/(R_i + R_j)$, where R_i and R_j are respectively the radius of curvature of profile i and j) . If there is no direct indication on the formula applicability range, one can refer to single articles and take the experiments conditions in terms of viscosity velocities and load as reference. Even though all the researchers provided different formulation for the friction coefficient, their observations in mixed condition (i.e. region B in the Stribeck curve) are quite similar and summarized below:

- The entraining lubricating fluid kinematic viscosity (ν_0) has a strong effect on the friction coefficient. They observed a rise in friction coefficient with reduced ν_0 apart from cases where there were small sliding speed and high rolling speed (i.e. surface speed sum V_Σ). The bigger the sliding speed the higher the effect of viscosity on friction coefficient.

- The friction coefficient lowers when rolling velocity V_Σ increases. This effect is more significant with reduced load, viscosity and sliding speed. The same behaviour can be observed for the sliding speed V_s.

- Increasing the load W (i.e the contact pressure) results in an higher friction coefficient.

Due to the experimental observation agreement in this regime of lubrication, the empirical formulas reported are similar from the physical quantity dependence viewpoint. Besides all the already mentioned physical quantities that influence the friction coefficient, Kelley & Lemanski propose,

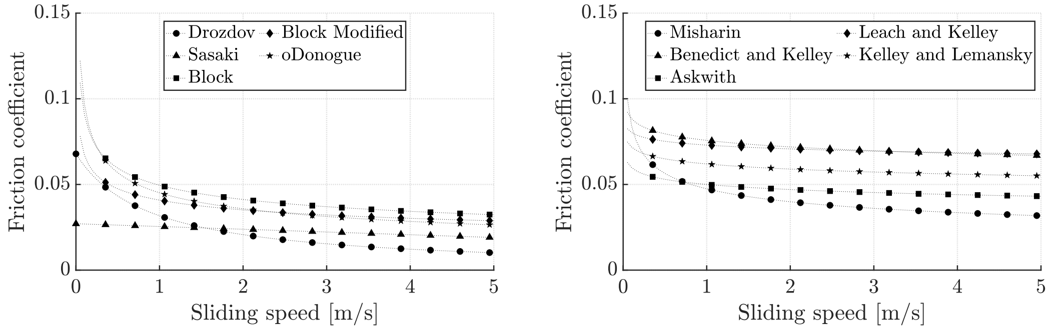

Figure 2. Friction coefficient varying sliding speed for different semi-empirical formulations.

Table 1. Friction coefficients under combined rolling and sliding

Author	Formula
Y.A. Misharin[1][29, 30]	$\mu = 0.325\left(\nu_0 V_\Sigma V_s\right)^{-0.25}$
Sasaki[2]*et al.* [31]	$\mu = k\left(\frac{\eta V_r^4}{W^{1.7}}\right)^{-0.12}$
H. Blok[3][29]	$\mu = 0.126\left(\eta_0 V_\Sigma^{0.85} V_s^{1.44} W^{-0.43}\right)^{-0.184}$
H. Blok[3][29]	$\mu = 0.118\left(\beta_0^{-0.16}\eta_0 V_\Sigma^{1.19}\, V_s^{1.22} W^{-0.44}\right)^{-0.178}$
Askwith *et al.*[4][32, 33]	$\mu = 0.0099\log_{10}\left(\frac{3.89\cdot 10^7 W}{\eta_0 V_s V_\Sigma^2}\right)$
Benedict & Kelley[4][24]	$\mu = 0.0127\log_{10}\left(\frac{3.17\cdot 10^8 W}{\eta_0 V_s V_\Sigma^2}\right)$
Leach & Kelley [4][34]	$\mu = -0.00738\log_{10}\left[1.936\cdot 10^{26}\left(\frac{V_s V_\Sigma^2}{\eta_0^7 W^{7.9}}\right)\right]$
Kelley & Lemanski[4][33]	$\mu = 0.0099\log_{10}\left(\frac{3.89\cdot 10^7 W}{\eta_0 V_s V_\Sigma^2\left[\left(R_i+R_j\right)/3\right]^2}\right)$
O'Donoghue & Cameron[5][35]	$\mu = \left(\frac{S+22}{35}\right)\frac{0.6}{\eta_0^{\frac{1}{8}} V_s^{\frac{1}{3}} V_\Sigma^{\frac{1}{6}} R^{\frac{1}{2}}}$
Drozdov & Gavrikov [6][36]	$\mu = \frac{1}{0.8\nu_0^{\frac{1}{2}} V_s + V_\Sigma\phi + 13.4}$

[1] Validity range $0.02 \le \mu \le 0.08$, ν_0 [cSt], V_s and V_Σ [m/s] .

[2] η_0 [kgf$\cdot$ s/cm^2], W [kgf/cm], $V_r = \frac{V_\Sigma}{2}$ [cm/s], $S = \frac{V_s}{V_r}$,

$$k = \begin{cases} 0.037 \text{ if } S = 0.31, \\ 0.026 \text{ if } S = 1.22 \end{cases} .$$

[3] η_0 [cP], $\beta_0 = 0.012\eta_0^{0.33}$ [$^\circ$C^{-1}], V_s and V_Σ [m/s], W [kgf/cm].

[4] V_s and V_Σ [inch/s], W [lb/in], η_0 [cP], R_i and R_j [inch].

[5] S: [μ inch], V_s and V_Σ [inch/s], R [inch].

[6] ν_0 [cSt], V_s and V_Σ [m/s] , $\phi = 0.47 - 0.13\cdot 10^{-4} p_{\max} - 0.4\cdot 10^{-3}\nu_0$, with $p_{\max}$, maximum contact pressure [kgf/cm^2]. $V_s \le 15$ m/s, $3 \le V_\Sigma \le 20$ m/s, $4 \le \eta_0 \le 500$ cSt, $4000 \le p_{\max} \le 20000$ kgf/cm^2.

without experimental evidence, a factor dependent on gear dimensions by the means of the radii of curvature. O'Donaghue & Cameron are the only authors among the ones herein reviewed that directly include the surface roughness effect in the friction coefficient formulation. The other investigators either do not discuss it or give corrective coefficients. Figure 2 reports comparative results of the friction coefficient computed through all different friction models as a function of the sliding speed.

4 MULTIBODY MODEL

A scotch-yoke mechanism with a straight guide, as depicted in Figure 3, has been chosen for testing different friction formulations. This mechanism is made up of four rigid bodies (crank, pin, slider and frame), two revolute joints, one prismatic joint and a non ideal joint with friction between pin and slider slot. The pin revolute joint is modelled with friction in order to avoid uncontrolled rotational speed of the pin itself, while all the other kinematic pairs are ideal and frictionless. The geometrical data and inertia parameters are summarized in Table 2, while all the other parameters are listed in Table 3. It is worth noting that the system taken as a case study has

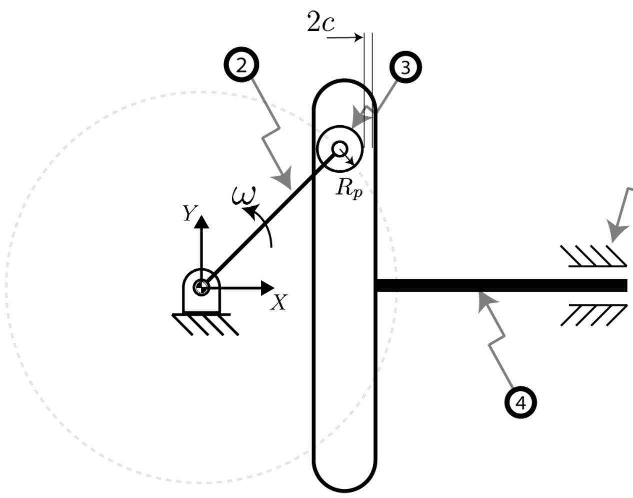

Figure 3. Scotch-Yoke mechanism with pin-slot non-ideal joint.

lubricated contacts with backlash and can not be reduced to a stationary system. It is possible to formulate the equations of motion for a dynamic system under holonomic constraints as a system of DAE. Indicated by q the position and orientation of all bodies the system can be expressed as:

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{\Psi}_q^T \\ \boldsymbol{\Psi}_q & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{Q} \\ \boldsymbol{\gamma} \end{Bmatrix} \tag{5}$$

where $\boldsymbol{M}$ is the system mass matrix, $\boldsymbol{\Psi}_q$ is the Jacobian matrix related to kinematic constraints, $\ddot{q}$ is the vector containing generalized accelerations, $\boldsymbol{\lambda}$is the Lagrange multiplier vector, $\boldsymbol{Q}$ is the vector of generalized forces and $\boldsymbol{\gamma}$ can be expressed as

$$\{\boldsymbol{\gamma}\} = -([\boldsymbol{\Psi}_q]\{\dot{\boldsymbol{q}}\})_q\{\dot{\boldsymbol{q}}\} - [\boldsymbol{\Psi}_{qt}]\{\dot{\boldsymbol{q}}\} - [\boldsymbol{\Psi}_{tt}] \tag{6}$$

From a numerical prospective, the multibody dynamic simulation is performed using the Baumgarte method [37] to ensure the stabilization of kinematic constraint violation. The contribution of contact pairs is accounted into the force vector $\boldsymbol{Q}$ rather than in the jacobian $\boldsymbol{\Psi}$ as the kinematic joints. The acceleration vector $\ddot{\boldsymbol{q}}$ and Lagrange multipliers $\boldsymbol{\lambda}$ are computed given the initial conditions for position and velocity. The resulting acceleration is integrated to obtain velocity and position at the next time step. The numerical integration is performed using the Matlab routine *ode45*, based on the explicit Runge-Kutta method. As it is already pointed out, the cylindrical contact force models available in literature are mostly implicit. In order to speed up the computation, using the approach employed in [17], the stiffness parameter K is derived from polynomial fitting while the dissipation component is evaluated according to the Equation 1.

Geometry		Member	Mass	Inertia
Crank length	50 mm	Crank	0.300 kg	3.8e2 kg·mm^2
Slot width	16 mm	Pin	0.025 kg	7.8e-1 kg·mm^2
Slot thickness	16 mm	Slider	0.300 kg	
Clearance	50 μm			

Table 2. Geometric and inertia parameters of the Scotch-Yoke mechanism.

Young's modulus	207 GPa	Baumgarte coefficients $[\alpha, \beta]$	[5,5]
Poisson ratio	0.3	Integration step size	variable
Restitution coefficient	0.9	Integration tolerance	1e-6
Dynamic viscosity	400 cP	Crank angular velocity ω	5000 rpm

Table 3. Data for dynamic simulation of the Scotch-Yoke mechanism.

5 RESULTS AND DISCUSSION

The outcomes of three non-ideal joint with clearance cases will be shown in this section: dry without friction, dry with constant friction ($\mu = 0.1$), and lubricated joint with friction estimated by the means of previously reviewed semi-empirical models. Generally, reaction force in joint with clearance exhibits peaks not observed in the ideal joint. These spikes are generated from impact and this can result in a chaotic and non-deterministic behaviour. However, in this particular case it can be observed a periodic trend in which the solution is not effected by initial conditions. For this reason, only results for one crank full rotation are presented. Firstly, a sensitivity analysis is conducted in order to highlight the effect on the interaction force of both the restitution coefficient c_r and the clearance c between pin and slot. The effect of clearance, as shown in Figure 4, is twofold. On the one hand, a larger clearance increases the impact speed by increasing the free flight time and reducing the damping effect due to hysteresis.

On the other hand, the motion will be much more irregular with continuous bouncing. As a consequence, with the clearance rise, the normal force shows higher peaks and the detachment between the surfaces happens more frequently, as shown on the left side of Figure 4. Furthermore, on the right side, the trajectory of the pin center in the slider reference system is plotted. To compare the different cases as the clearance varies, the relative X coordinate has been normalized.

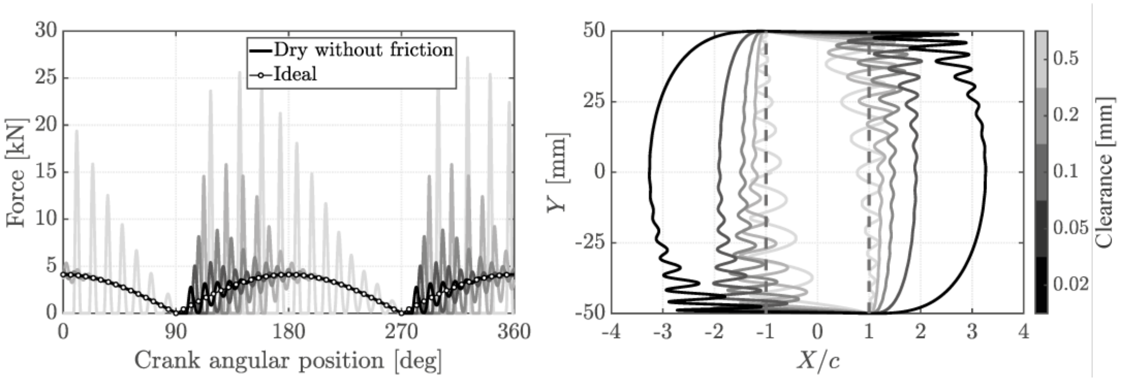

Figure 4. Clearance c sensitivity: normal contact force (left); pin center trajectory (right).

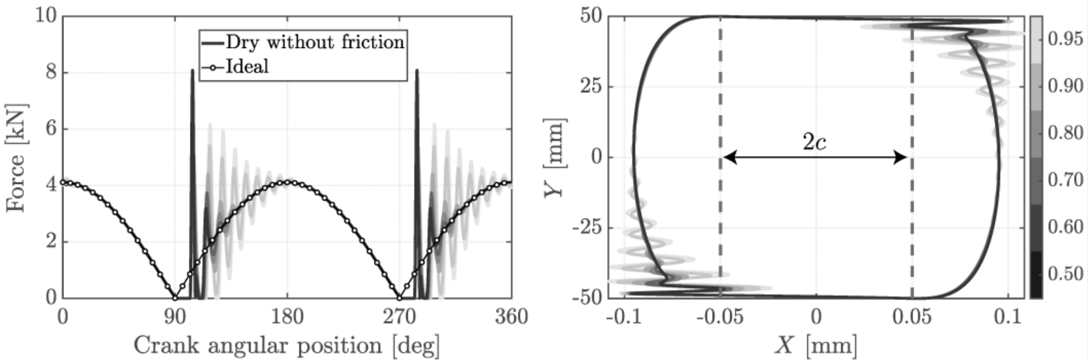

Figure 5. Restitution coefficient c_r sensitivity: normal contact force (left); pin center trajectory (right).

Instead, the restitution coefficient, as it is possible to observe from Figure 5, slightly effects the oscillation frequency while has a strong influence on the oscillation amplitude.

In this particular configuration, since the effects are decoupled, the normal force, illustrated in Figure 6, is not influenced by the friction force. Different friction models have been tested and their value are plotted versus the crank angle (Figure 7-8). Even though the different friction formulations provide similar trend, their magnitude are quite different between one and other. In particular, Misharin and Drozdov models produce extremely low values of friction coefficient during high sliding speed phases and give feasible values only when the sliding speed lowers (i.e pin motion inversion at $\vartheta_{crank} = \frac{\pi}{2} + k\pi$ with $k = 1, 2, ..., n$). Furthermore, it should be noted that only the Leach and Kelley model has a distinctive trend. This fact can be explained with the strong load dependence that these authors propose in their empirical formulation causing the friction coefficient to assume the normal force trend instead of showing a predominant sliding speed effect.

Referring to the Figure 9 the trend of the driving torque is displayed in three cases: ideal case, non-ideal kinematic joint without friction, and with friction using Benedict & Kelley model. In the non-ideal joint case, the torque exhibits a dynamic behavior due to impacts that occur on the contact surfaces. Such events happen only during the transition between the acceleration and deceleration

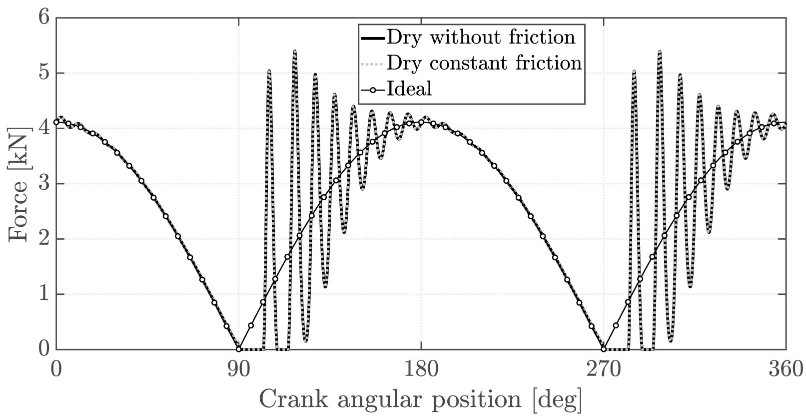

Figure 6. Comparison of normal contact force in non-ideal joint with reaction force of ideal joint.

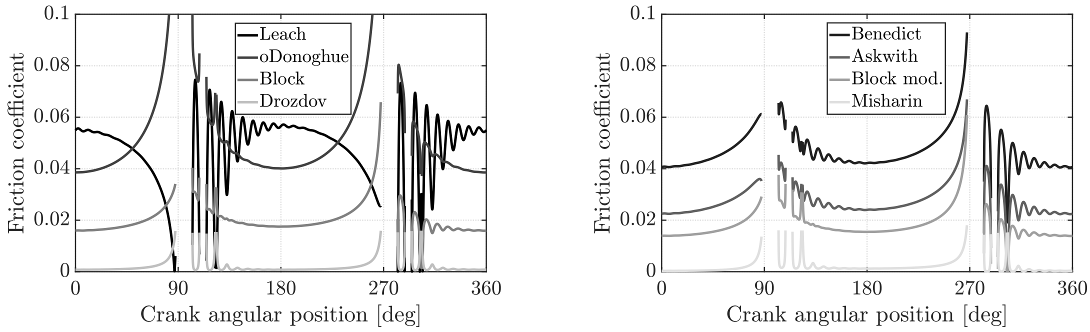

Figure 7. Friction coefficient for different semi-empirical formulations.

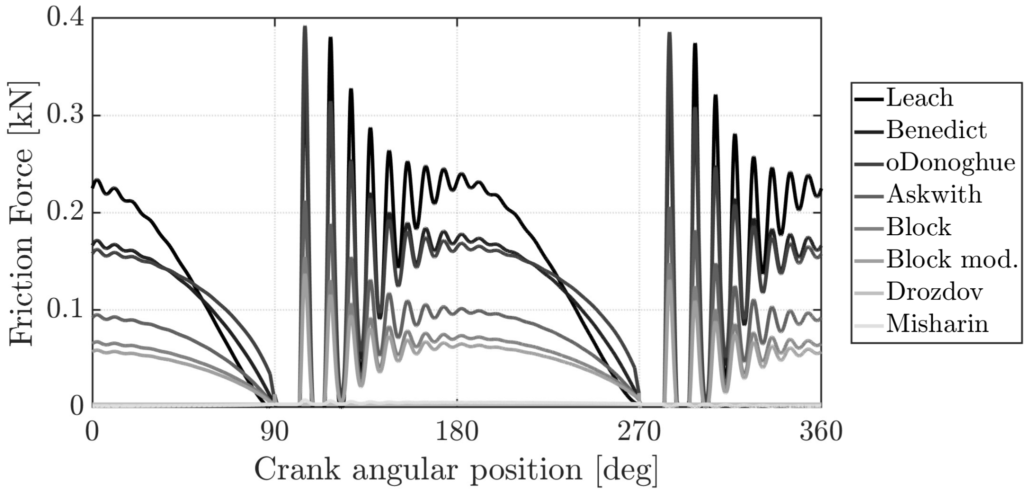

Figure 8. Friction force for different semi-empirical formulations.

phase of the slider, where a free-flight period of the pin and a change in the contact profile of the slot are observed. Conversely, the transition between the deceleration and acceleration phase occurs continuously, and no sudden variations of the torque are observed. Having said that, for a constant crank speed, two phases can be observed in a crank rotation period of 180 degrees. In the first phase (i.e. between 0 and 90 degrees), the slider accelerates and therefore the driving torque is positive; the ideal case and the contact case without friction have little difference, while the driving torque slightly increases when friction is taken into account. In the second phase (i.e between 90

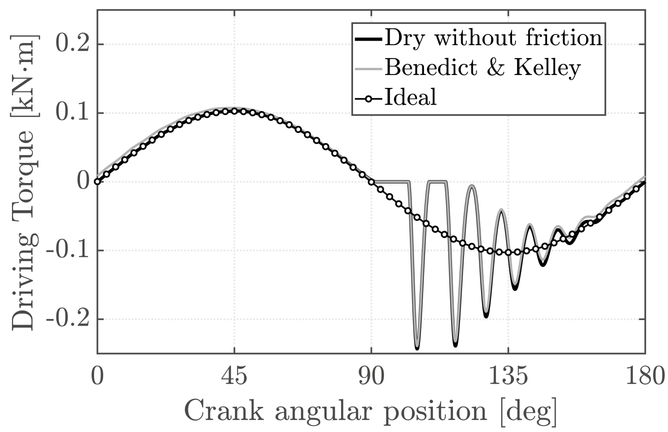

Figure 9. Comparison of driving torque between ideal and non-ideal cases.

and 180 degrees) the slider decelerates, so the driving torque is negative. In this case, the friction force tends to slow down the slider as a result of the retrograde motion that is observed during this phase. Because of this, the driving torque in the lubricated case is lower in absolute terms than the dry without friction one. For any considered condition the computation time, using Matlab to perform the multibody simulation, is reported in Table 4. A remarkable increase of run-time can be observed with the introduction of a non-ideal joint. In contrast, the introduction of friction models has not a relevant consequence on the run-time itself and .

	Ideal joint	**Dry without friction**	**Dry with constant friction**
Runtime [s]	0.5	4.3	4.3

	Lubricated models						
	Leach	Benedict	O'Donoguhue	Askwith	Block	Drozdov	Misharin
Runtime [s]	5.0	5.0	5.4	5.2	5.1	4.8	4.7

Table 4. Computation time (average on 10 runs) to solve 4 full crank rotation with processor Intel I7-9750H, 2.60 GHz

6 CONCLUSIONS

In this work, semi-empirical tools for embedding friction into non stationary mechanical systems with unilateral contacts under sliding and rolling conditions have been suggested. These formulations are applied within multibody simulations where non-ideal kinematic pairs with clearance are present, and their applicability for calculating the friction force is demonstrated. These formulations are derived for the contact between gear teeth and are generally valid models for conditions of partial elastohydrodynamic lubrication. Therefore, a scotch-yoke mechanism is chosen as the linkage to be tested, in which the contact between the pin and the slot is counterformal. From the results, it can be concluded that the Benedict & Kelley, O'Donoghue & Cameron, Askwith *et al.* and Block models are consistent and show comparable trends. Conversely, the Leach & Kelley model exhibits a significant dependence on the load. Lastly, the Drozdov and Misharin models return very low friction coefficients for high sliding velocities. Using these models in multibody simulations, the computational cost does not significantly increase compared to imposing a constant friction coefficient. Another advantage of these models is that the system exhibits a non-stiff behavior, unlike hydrodynamic models where adjusting simulation parameters is necessary to avoid numerical errors. In conclusion, the analytical comparisons performed here and all the observations reported are specific to the situation under study and may not apply to other situations.

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