# End Effects in Spatial Beam Elements: Comparing the Variational Asymptotic Method and the Eigenwarping Method 

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#### Abstract

Spatial beam elements can be used for efficient simulation and optimisation of flexure mechanisms undergoing large deflections. However, for flexures with low slenderness ratio, end effects can become relevant. These effects can be captured by introducing additional warping fields in the formulation. In this work, we compare the warping fields generated by the Variational Asymptotic Method (VAM) to the warping fields generated by a recently developed method, referred to as the Eigenwarping method (EWM). VAM constructs an expansion of the cross sectional warping, where it is shown that each term has an asymptotically smaller contribution to the internal elastic energy in terms of the slenderness ratio $h / L$. By truncating the expansion at a given order, an accurate representation of the beam behaviour is obtained for only a small number of required warping coordinates. The EWM considers the decomposition of the cross sectional constitutive behaviour in non-decaying fields associated with an elastic line and decaying cross sectional warping fields. The resulting set of decaying warping fields can be truncated based on the associated decay length. In this paper, an alternate interpretation of VAM is presented that allows better comparison to the EWM. The warping fields obtained from both methods are compared to each other and to classical results. Additionally, a beam element with warping fields is introduced. Using the beam element, simulation results obtained with the warping fields obtained from VAM and EWM can be compared. The case of a notched leaf flexure is introduced, which contains a discontinuity in the cross section. Both methods can describe the behaviour in the continuum and at fixed boundaries with only a small number of additional warping fields. However, a large number of warping fields is required to accurately describe the behaviour at the notch boundary.


Keywords: Beam element, Warping, Boundary conditions.

## 1 INTRODUCTION

Beam elements are widely used for the simulation of flexure mechanisms undergoing large deflections. The low number of associated degrees of freedom results in highly efficient simulations for complex mechanisms. The assumptions behind the beam elements are usually based on the slenderness of the beam. That is, the beam has a length $L$ that is much greater than the dimensions of the cross section $h$. For slender beams, the end effects become negligible. However, for shorter beams, these effects can have a significant influence on the stiffness and stress. Similarly, they can become significant when the cross section of the beam discontinuously changes, for example due to local reinforcement or notches in leaf flexures [1, 2].
Several methods exist for modelling the end effects in spatial beam elements. Yu and Hodges applied the Variational Asymptotic Method (VAM) to approximate the internal energy of a spatial beam section up to a specific slenderness order $(h / L)^{n}$ [3]. The results are a generalized Timoshenko [4] and Vlasov model [5] for spatial beams with arbitrary anisotropic cross section. Additionally, the warping field of the cross section is obtained as a function of the traditional beam curvatures.

Instead of using the variational asymptotic method, one may also construct an eigenvalue problem associated with the cross sectional Hamiltonian, as demonstrated by Han and Bauchau [6]. This results in eigenvectors associated with the deformation of the cross section. As far as the authors are aware, there is no consistent name for this method. Inspired by the terminology in earlier work of Bauchau [7], we will refer to the eigenvectors as Eigenwarpings and to the overall method as the Eigenwarping Method (EWM).

Both VAM and EWM yield a decomposition of the cross-sectional displacement field. By retaining only the dominant terms, an accurate representation of the beam behaviour can be obtained. Furthermore, the low number of associated coordinates yield relatively low computational effort. In the case of VAM, the truncation is based on the slenderness order of the warping terms $(h / L)^{n}$. For EWM, the truncation is based on the decay length of the Eigenwarpings.

In this work, we show that the VAM and EWM are both based on locally linear behaviour of the cross section. Exploiting the locally linear behaviour, we construct a simple linear beam element that incorporates a truncated set of warping fields, similar to e.g. [8]. Using this beam element, we evaluate the trade-off between accuracy and computational cost, associated with the number of retained warping fields for both methods. In particular, we consider the case of a notched leaf flexure. The notched leaf flexure contains a discontinuity in the cross-section, resulting in significant end effects. The performance of VAM and EWM is compared to a 3D finite element model in COMSOL.

In the next section, VAM and EWM will be introduced and the resulting warping fields will be compared. In section 4, a simple beam element with warping coordinates is introduced. Thereafter, in section 5, performance of the warping fields is discussed for a notched leaf flexure case. Finally, the conclusions and further research are discussed.

## 2 CROSS SECTION ANALYSIS

Han [6] and Hodges [3] take a different approach in deriving the behaviour of the cross section from spatial beams. Here we show the similarity between the two methods and establish a baseline formulation that will allow better comparison between both methods.
Han introduces a local Frenet-Serret frame to obtain the local strain $\boldsymbol{\gamma}$ and cross sectional strain energy $U$

$$
\begin{align*}
& U(s)=\iint \boldsymbol{\gamma}^{T} \mathscr{D} \boldsymbol{\gamma} d A  \tag{1}\\
& \boldsymbol{\gamma}(s, y, z)=\Gamma \boldsymbol{w}(s, y, z) \tag{2}
\end{align*}
$$

Here, $\mathscr{D}$ contains the consitituve behaviour, $\Gamma$ represents the small strain operator, $d A=d y d z$ and $\boldsymbol{w}$ the local deflection in the neighbourhood of the cross section. This may be interpreted as taking a small slice of the beam and analysing it in a local frame of reference. In this local frame the displacements are small and the slice behaves as if it were a linear beam.
Hodges takes a different approach, leading to the same expression for the internal energy. However, a different expression for $\boldsymbol{\gamma}$ and an additional constraint are obtained, which are given by

$$
\begin{align*}
\boldsymbol{\gamma}(s, y, z) & =L(y, z) \overline{\boldsymbol{\gamma}}(s)+\Gamma \boldsymbol{v}(s, y, z)  \tag{3}\\
0 & =\iint D^{T}(y, z) \boldsymbol{v}(s, y, z) d y d z \tag{4}
\end{align*}
$$

where $\overline{\boldsymbol{\gamma}}=\left[\begin{array}{llll}\bar{\gamma}_{x} & \bar{\kappa}_{x} & \bar{\kappa}_{y} & \kappa_{z}\end{array}\right]^{T}$ contains the local stretch and curvature associated with the elastic line of a spatial beam, $\boldsymbol{v}(s, y, z)$ is the warping of the cross section and $E(y, z)$ and $D(y, z)$ are given later on in Eq. (6).
The term $L(y, z) \overline{\boldsymbol{\gamma}}(s)$ in Eq. (3) results from the deformation of the elastic line. This term can also
be interpreted as the strain resulting from an Euler-Bernoulli displacement field:

$$
\begin{equation*}
L(y, z) \overline{\boldsymbol{\gamma}}(s)=\Gamma\left(D(y, z) \overline{\boldsymbol{u}}(s)+\tilde{D}(y, z) \overline{\boldsymbol{u}}^{\prime}(s)\right) . \tag{5}
\end{equation*}
$$

where $\overline{\boldsymbol{u}}(s)=\left[\begin{array}{llll}\bar{u} & \bar{v} & \bar{w} & \bar{\theta}\end{array}\right]^{T}$ contains the typical beam coordinates for small deformations, denotes the spanwise derivative $\partial \square / \partial s$ and the matrices $D, \tilde{D}$ and $E$ are given by

$$
D=\left[\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{6}\\
0 & 1 & 0 & -z \\
0 & 0 & 1 & y
\end{array}\right] ; \quad \tilde{D}=\left[\begin{array}{cccc}
0 & -y & -z & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right] ; \quad L^{T}=\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & y & -z \\
-z & 0 & 0 & 0 & 0 & 0 \\
y & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

As a consequence of Eq. (3), Eq. (5) and Eq. (2), it can be concluded that the warping $\boldsymbol{v}$, used by Hodges, is related to the warping $\boldsymbol{w}$, used by Han, using the following relation

$$
\begin{equation*}
\boldsymbol{w}=\boldsymbol{v}+D(y, z) \overline{\boldsymbol{u}}(s)+\tilde{D}(y, z) \boldsymbol{u}^{\prime}(s) \tag{7}
\end{equation*}
$$

Conversely, $\overline{\boldsymbol{u}}$ (and by Eq. (7) also $\boldsymbol{v}$ ) may also be constructed from $\boldsymbol{w}$ by employing the constraint given in Eq. (4), resulting in

$$
\begin{align*}
0 & =\iint D^{T}(y, z)\left(\boldsymbol{w}-D(y, z) \overline{\boldsymbol{u}}(s)+\tilde{D}(y, z) \boldsymbol{u}^{\prime}(s)\right) d A  \tag{8}\\
& =\iint D^{T} \boldsymbol{w} d A-\iint D^{T} D d A \overline{\boldsymbol{u}}-\iint D^{T} \tilde{D} d A \boldsymbol{u}^{\prime}  \tag{9}\\
\rightarrow \iint D^{T} D d A \overline{\boldsymbol{u}} & =\iint D^{T} \boldsymbol{w} d A \tag{10}
\end{align*}
$$

where the last equation only holds if the origin of the cross section is located in the geometric center, as this will result in $\iint D^{T} \tilde{D} d A=0$. Since the strain formulations of Han and Hodges are equivalent, we continue with the strain relation given in Eq. (2), keeping in mind that we can easily reconstruct $\overline{\boldsymbol{u}}$ from $\boldsymbol{w}$ using Eq. (10).
Both Han and Hodges proceed by partially discretising the cross section. Here, we show the discretisation for $\boldsymbol{w}$,

$$
\begin{equation*}
\boldsymbol{w}(s, y, z)=N(y, z) \overline{\boldsymbol{w}}(s) \tag{11}
\end{equation*}
$$

Typically, piecewise quadratic basis functions are used. Since $D$ and $\tilde{D}$ are linear in $y$ and $z$, the exact same discretisation can be used for $\boldsymbol{v}$. Substituting this in Eq. (2) yields, as detailed in [6],

$$
\begin{equation*}
\boldsymbol{\gamma}=\Gamma(N(y, z) \boldsymbol{w}(s))=B(y, z) \overline{\boldsymbol{w}}(s)+G(y, z) \overline{\boldsymbol{w}}^{\prime}(s) \tag{12}
\end{equation*}
$$

It follows that the cross sectional internal energy may be written as:

$$
U(s)=\left[\begin{array}{c}
\overline{\boldsymbol{w}}  \tag{13}\\
\overline{\boldsymbol{w}}^{\prime}
\end{array}\right]^{T} \iint\left[\begin{array}{cc}
B^{T} \mathscr{D} B & B^{T} \mathscr{D} G \\
G^{T} \mathscr{D} B & G^{T} \mathscr{D} G
\end{array}\right] d A\left[\begin{array}{c}
\overline{\boldsymbol{w}}^{\overline{\boldsymbol{w}}^{\prime}}
\end{array}\right]=\left[\begin{array}{c}
\overline{\boldsymbol{w}} \\
\overline{\boldsymbol{w}}^{\prime}
\end{array}\right]^{T}\left[\begin{array}{cc}
E & C \\
C^{T} & M
\end{array}\right]\left[\begin{array}{c}
\overline{\boldsymbol{w}} \\
\overline{\boldsymbol{w}}^{\prime}
\end{array}\right]
$$

Introducing the discretisation into Eq. (10), $\overline{\boldsymbol{u}}$ can be expressed as:

$$
\begin{equation*}
\overline{\boldsymbol{u}}(s)=\left(\iint D^{T} D d A\right)^{-1} \iint D^{T} N d A \overline{\boldsymbol{w}}=\Psi \overline{\boldsymbol{w}}(s) \tag{14}
\end{equation*}
$$

In order to retain $\overline{\boldsymbol{u}}(s)$ as explicit variables, $U(s)$ can be modified with the Lagrange multipliers $\lambda$.

$$
U(s)=\frac{1}{2}\left[\begin{array}{c}
\overline{\boldsymbol{w}}  \tag{15}\\
\overline{\boldsymbol{w}}^{\prime}
\end{array}\right]^{T}\left[\begin{array}{cc}
E & C \\
C^{T} & M
\end{array}\right]\left[\begin{array}{c}
\overline{\boldsymbol{w}} \\
\overline{\boldsymbol{w}}^{\prime}
\end{array}\right]-\boldsymbol{\lambda}^{T}(\overline{\boldsymbol{u}}-\Psi \overline{\boldsymbol{w}})
$$

Having established the discretised version of the sectional internal energy, the VAM warping expansion and the Eigenwarpings will be introduced in the next two sections respectively.

### 2.1 VARIATIONAL ASYMPTOTIC METHOD

The variational method is based on the recursive perturbation of the warping field [3]. By using $\mathscr{O}\left(\overline{\boldsymbol{w}}^{\prime}\right)=(h / L) \mathscr{O}(\overline{\boldsymbol{w}})$, each subsequent perturbation is used to obtain an internal energy up to a higher order. Carefully following the procedure results in the following expansion for $\overline{\boldsymbol{w}}$ and $\boldsymbol{\lambda}$ :

$$
\left[\begin{array}{l}
\overline{\boldsymbol{w}}  \tag{16}\\
\lambda
\end{array}\right]=\left[\begin{array}{l}
\bar{w}_{0} \\
\boldsymbol{\lambda}_{0}
\end{array}\right]+\left[\begin{array}{l}
\bar{w}_{1} \\
\lambda_{1}
\end{array}\right]+\left[\begin{array}{l}
\overline{\boldsymbol{w}}_{2} \\
\boldsymbol{\lambda}_{2}
\end{array}\right]+\mathscr{O}(h / L)^{3},
$$

with the following relations for the first two terms:

$$
\begin{align*}
& {\left[\begin{array}{l}
\overline{\boldsymbol{w}}_{0} \\
\boldsymbol{\lambda}_{0}
\end{array}\right]=\left[\begin{array}{ll}
E & \Psi^{T} \\
\Psi & 0
\end{array}\right]^{-1}\left[\begin{array}{l}
0 \\
I
\end{array}\right] \overline{\boldsymbol{u}}=\left[\begin{array}{l}
s_{w_{0}} \\
s_{\lambda_{0}}
\end{array}\right] \overline{\boldsymbol{u}}}  \tag{17}\\
& {\left[\begin{array}{l}
\overline{\boldsymbol{w}}_{1} \\
\boldsymbol{\lambda}_{1}
\end{array}\right]=-\left[\begin{array}{cc}
E & \Psi^{T} \\
\Psi & 0
\end{array}\right]^{-1}\left[\begin{array}{c}
C-C^{T} \\
0
\end{array}\right] \overline{\boldsymbol{w}}_{0}^{\prime}=\left[\begin{array}{l}
s_{w_{1}} \\
s_{\lambda_{1}}
\end{array}\right] \overline{\boldsymbol{u}}^{\prime}} \tag{18}
\end{align*}
$$

For $n \geq 2$, the following recursive relation is obtained.

$$
\left[\begin{array}{c}
\overline{\boldsymbol{w}}_{n+1}  \tag{19}\\
\lambda_{n+1}
\end{array}\right]=-\left[\begin{array}{cc}
E & \Psi^{T} \\
\Psi & 0
\end{array}\right]^{-1}\left(\left[\begin{array}{c}
C-C^{T} \\
0
\end{array}\right] \overline{\boldsymbol{w}}_{n}^{\prime}-\left[\begin{array}{c}
M \\
0
\end{array}\right] \overline{\boldsymbol{w}}_{n-1}^{\prime \prime}\right)=\left[\begin{array}{l}
s_{w_{n}} \\
s_{\lambda_{n}}
\end{array}\right] \overline{\boldsymbol{u}}^{(n)}
$$

If the original derivation is followed, a series expansion of $\boldsymbol{v}$ and $\boldsymbol{\lambda}$ in terms of $\overline{\boldsymbol{\gamma}}$ is obtained. The series above is equivalent. However, since $\overline{\boldsymbol{\gamma}}=\left[\begin{array}{llll}u^{\prime} & \theta^{\prime} & v^{\prime \prime} & w^{\prime \prime}\end{array}\right]^{T}$, the terms representing $u, v, v^{\prime}, w, w^{\prime}$ and $\theta$ are excluded in the original derivation. However, as can be seen in Eq. (7), this is compensated by the difference in definition between $\boldsymbol{v}$ and $\boldsymbol{w}$.
Since the asymptotic expansion process to obtain Eqs. (16-19) can be quite involved, we would instead like to interpret the VAM solution as an approximate solution to the following DAE. This DAE can be obtained by requiring Eq. (15) to be stationary with respect to $\boldsymbol{w}$ and $\boldsymbol{\lambda}$, yielding

$$
\begin{gather*}
{\left[\begin{array}{cc}
E & \Psi^{T} \\
\Psi & 0
\end{array}\right]\left[\begin{array}{l}
\overline{\boldsymbol{w}} \\
\boldsymbol{\lambda}
\end{array}\right]+\left[\begin{array}{cc}
C-C^{T} & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{l}
\overline{\boldsymbol{w}} \\
\boldsymbol{\lambda}
\end{array}\right]^{\prime}+\left[\begin{array}{cc}
-M & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{l}
\overline{\boldsymbol{w}} \\
\boldsymbol{\lambda}
\end{array}\right]^{\prime \prime}=\left[\begin{array}{l}
0 \\
I
\end{array}\right] \overline{\boldsymbol{u}}}  \tag{20}\\
\mathscr{A}  \tag{21}\\
{\left[\begin{array}{l}
\overline{\boldsymbol{w}} \\
\boldsymbol{\lambda}
\end{array}\right]+\mathscr{B}\left[\begin{array}{l}
\overline{\boldsymbol{w}} \\
\boldsymbol{\lambda}
\end{array}\right]^{\prime}+\mathscr{C}\left[\begin{array}{l}
\overline{\boldsymbol{w}} \\
\boldsymbol{\lambda}
\end{array}\right]^{\prime \prime}=\left[\begin{array}{l}
0 \\
I
\end{array}\right] \overline{\boldsymbol{u}} .}
\end{gather*}
$$

Inspired by VAM, we introduce the following trial solution

$$
\left[\begin{array}{l}
\boldsymbol{w}  \tag{22}\\
\boldsymbol{\lambda}
\end{array}\right]=\left[\begin{array}{l}
s_{w_{0}} \\
s_{\lambda_{0}}
\end{array}\right] \overline{\boldsymbol{u}}+\left[\begin{array}{l}
s_{w_{1}} \\
s_{\lambda_{1}}
\end{array}\right] \overline{\boldsymbol{u}}^{\prime}+\left[\begin{array}{l}
s_{w_{2}} \\
s_{\lambda_{2}}
\end{array}\right] \overline{\boldsymbol{u}}^{\prime \prime} \ldots=\left[\begin{array}{llll}
s_{0} & s_{1} & s_{2} & \ldots
\end{array}\right]\left[\begin{array}{c}
\overline{\boldsymbol{u}} \\
\overline{\boldsymbol{u}}^{\prime} \\
\overline{\boldsymbol{u}}^{\prime \prime} \\
\vdots
\end{array}\right]
$$

Substituting this in Eq. (20) and rearranging yields:

$$
\left.\left(\begin{array}{ccc}
\mathscr{A}\left[\begin{array}{llll}
s_{0} & s_{1} & s_{2} & \ldots
\end{array}\right)  \tag{23}\\
+\mathscr{B}[0 & s_{0} & s_{1}
\end{array} \ldots\right]\right]\left[\begin{array}{c}
\overline{\boldsymbol{u}} \\
+\mathscr{C}\left[\begin{array}{ccc}
0 & 0 & s_{0}
\end{array} \ldots\right]
\end{array}\right)\left[\begin{array}{ccc}
I & 0 & \ldots \\
\overline{\boldsymbol{u}}^{\prime} \\
\overline{\boldsymbol{u}}^{\prime \prime} \\
\vdots & 0 & \ldots
\end{array}\right]\left[\begin{array}{c}
\overline{\boldsymbol{u}} \\
\overline{\boldsymbol{u}}^{\prime} \\
\overline{\boldsymbol{u}}^{\prime \prime} \\
\vdots
\end{array}\right]
$$

Since the two sides should be equal, we can equate term by term to obtain the same warping fields as given in Eqs. (16-19). Hence, the warping fields obtained using VAM may be interpreted as a particular solution of the DAE given in Eq. (20).
Contrary to Hodges we have not explicitly considered the order $(h / L)^{n}$ of each of the terms. However, each of the subsequent warping fields $\boldsymbol{w}_{i}$ is asymptotically smaller than the previous. Therefore, a reasonable approximation of the continuum can be obtained by truncating the series after a certain amount of terms.

### 2.2 EIGENWARPINGS

The internal energy given in Eq. (13) can be rewritten into a Hamiltonian by taking the Legendre transform, where the dual variable $\bar{t}=\partial U / \partial \bar{w}^{\prime}$ is introduced. $\bar{t}$ may be interpreted as the surface traction on the cross section. This yields the Hamiltonian

$$
H=\left[\begin{array}{c}
\overline{\boldsymbol{w}}  \tag{24}\\
\overline{\boldsymbol{t}}
\end{array}\right]^{T}\left[\begin{array}{cc}
I & C \\
0 & M
\end{array}\right]^{-1}\left[\begin{array}{cc}
-E & 0 \\
-C^{T} & I
\end{array}\right]\left[\begin{array}{c}
\overline{\boldsymbol{w}} \\
\overline{\boldsymbol{t}}
\end{array}\right]=\left[\begin{array}{c}
\overline{\boldsymbol{w}} \\
\overline{\boldsymbol{t}}
\end{array}\right]^{T}\left[\begin{array}{cc}
E & \hat{C} \\
\hat{C}^{T} & \hat{M}
\end{array}\right]\left[\begin{array}{c}
\overline{\boldsymbol{w}} \\
\overline{\boldsymbol{t}}
\end{array}\right],
$$

with the corresponding differential equation:

$$
\left[\begin{array}{c}
\overline{\boldsymbol{w}}  \tag{25}\\
\overline{\boldsymbol{t}}
\end{array}\right]^{\prime}=J\left[\begin{array}{cc}
E & \hat{C} \\
\hat{C}^{T} & \hat{M}
\end{array}\right]\left[\begin{array}{c}
\overline{\boldsymbol{w}} \\
\overline{\boldsymbol{t}}
\end{array}\right]=F\left[\begin{array}{c}
\overline{\boldsymbol{w}} \\
\overline{\boldsymbol{t}}
\end{array}\right] ; \quad \text { with } J=\left[\begin{array}{cc}
0 & I \\
-I & 0
\end{array}\right]
$$

This equation poses an eigenvalue problem. Han shows that there are 12 repeated eigenvalues with value 0 , which can be associated with generalised eigenvectors corresponding to $u, u^{\prime}, \theta, \theta^{\prime}, v, v^{\prime}, v^{\prime \prime}$, $v^{\prime \prime \prime}, w, w^{\prime}, w^{\prime \prime}$ and $w^{\prime \prime \prime}$. The remaining eigenvalues and eigenvectors represent exponentially decaying modes. It follows that there is a symplectic transformation $V$, which contains the eigenvectors, that diagonalises $F$. Due to the Hamiltonian structure, each eigenvalue $\mu$ has a corresponding eigenvalue $-\mu$. This can be interpreted as a decaying mode for both the left and the right ends of the beam. Furthermore, if $\mu$ is complex, then the complex conjugates of $\mu$ and $-\mu$ are also eigenvalues.
Here, a slight modification to the approach used by Han is introduced. We consider the following equivalent generalised eigenvalue problem to compute the eigenvectors.

$$
\left[\begin{array}{cc}
C & -M  \tag{26}\\
I & 0
\end{array}\right]\left[\begin{array}{c}
\overline{\boldsymbol{w}} \\
\overline{\boldsymbol{t}}
\end{array}\right]^{\prime}=\left[\begin{array}{cc}
-E & 0 \\
-C^{T} & I
\end{array}\right]\left[\begin{array}{c}
\overline{\boldsymbol{w}} \\
\overline{\boldsymbol{t}}
\end{array}\right]
$$

This form preserves the sparsity of $E, C$ and $M$, which is numerically beneficial if only a reduced set of Eigenvectors with a long decay length is computed. Since the neglected eigenwarpings have a very short decay length, their influence on the beviour of the continuum is negligible.

## 3 COMPARING VAM AND EWM

In the previous section we outlined that VAM and EWM are both based on the same sectional internal energy. VAM yielded a series of warping fields related to the traditional beam variables $\overline{\boldsymbol{u}}$ and its derivatives. Similar to VAM, the EWM yielded a series of warping fields. However, each of the resulting warping fields now has an associated decay rate.
In the remainder of this section we consider an isotropic rectangular section and compare the warping fields obtained using VAM to those obtained using EWM. Furthermore, similarities with classical results from literature are discussed.

### 3.1 A RECTANGULAR SECTION

In this section an isotropic rectangular cross section is considered. The section has dimensions $w=1 \mathrm{~m}, h=0.1 \mathrm{~m}$ and a Poisson ratio of $v=0.3$. The Young's modulus of the material is not relevant, since it only scales the section internal energy.
Figure 1 shows the warping fields obtained using the described VAM procedure and figure 2 shows the warping fields obtained with the EWM procedure with the associated eigenvalue $\mu$.
We would like to point out several classically known effects: The anticlastic curvature ( $v^{\prime \prime}$ and $w^{\prime \prime}$ in Fig. 1; and 2 and 3 in Fig. 2), shear warping ( $v^{\prime \prime \prime}, w^{\prime \prime \prime}$ and 5,6), Saint-Venant torsional warping ( $\theta^{\prime}$ and 4) and the Poisson effect ( $u^{\prime}$ and 1 ).
Curiously, as also pointed out in [6], the shear terms ( $\left.v^{\prime \prime \prime}, w^{\prime \prime \prime}\right)$ are of $\mathscr{O}(h / L)$ according to VAM, however their decay rate is zero according to the $\operatorname{EWM}(5,6)$. Furthermore, a slow decay solution,


Figure 1. The first 16 cross sectional warping fields $\boldsymbol{w}_{i}$ obtained using the VAM.
which is an approximate combination of anticlastic curvature and shear, is also present in EWM $(7,8)$. As of now, we have found no intuitive explanation for this difference.
The EWM warping fields $(9,10)$ are related to a generalised version of the well known Vlasov torsion theory. This torsion theory can be reconstructed from VAM, as outlined in [5]. The resulting decay rate is associated with the warping fields $\theta^{\prime}, \theta^{\prime \prime}$ and $\theta^{\prime \prime \prime}$ and for this section results in a $37 \%$ decay length of 0.2395 m for VAM, which differs only slightly from the decay length of $1 / \mu=0.2436 \mathrm{~m}$ obtained by the EWM.
Other similarities between the warping fields also do exist. For example $u^{\prime \prime}, u^{\prime \prime \prime}, u^{\prime \prime \prime \prime}$ and $u^{\prime \prime \prime \prime \prime}$ are approximately linear combinations of $(11,12,13,14)$.

## 4 BEAM ELEMENT WITH WARPING

In the previous section we have compared the warping fields produced by VAM and EWM for a rectangular cross section. In this section a simple beam element is derived based on a truncated set of warping fields from either EWM or VAM. In the next section, the beam element will be used to determine the number of warping fields required to accurately describe the end effects occurring in a notched leaf flexure.
As shown in section 2, locally the behaviour of a spatial beam is linear. In order to keep the evaluation of the warping simple, we opt to evaluate their performance here within a linear beam framework. The linearised cross section behaviour shown here can be extended to a beam element for arbitrary rigid body body motion by considering the element in a co-rotational frame of reference. However, the extension will not be discussed here.
Consider the following expansion of the sectional displacement field

$$
\begin{equation*}
\overline{\boldsymbol{w}}(x)=\boldsymbol{\Phi} \overline{\boldsymbol{\alpha}}(x) . \tag{27}
\end{equation*}
$$



Figure 2. The first 16 cross sectional warping fields $\boldsymbol{w}_{i}$ obtained using the EWM. $\mu_{i}$ denotes the eigenvalue corresponding to the depicted Eigenwarping.

Both VAM and EWM yield as the first six warping fields the rigid body translation and rotation of the cross section, denoted by $\bar{\alpha}_{1-6}=\left[\begin{array}{llllll}\bar{u}_{x} & \bar{u}_{y} & \bar{u}_{z} & \bar{\phi}_{x} & \bar{\phi}_{y} & \bar{\phi}_{z}\end{array}\right]^{T}$. The remainder of the fields are taken as the warping fields shown in Figure 1 for VAM and Figure 2 for EWM. In VAM, the expansion can be truncated at a certain order $(h / L)^{n}$, whereas in the case of the EWM, the fields can be truncated based on the decay length $\operatorname{real}(\mu)<\sigma_{n}$.

Using the constructed basis $\Phi$ for the displacement field, we project the internal energy

$$
U=\frac{1}{2} \int\left[\begin{array}{c}
\overline{\boldsymbol{\alpha}}  \tag{28}\\
\overline{\boldsymbol{\alpha}}^{\prime}
\end{array}\right]^{T}\left[\begin{array}{cc}
\Phi & 0 \\
0 & \Phi
\end{array}\right]^{T}\left[\begin{array}{cc}
E & C \\
C^{T} & M
\end{array}\right]\left[\begin{array}{cc}
\Phi & 0 \\
0 & \Phi
\end{array}\right]\left[\begin{array}{c}
\overline{\boldsymbol{\alpha}} \\
\overline{\boldsymbol{\alpha}}^{\prime}
\end{array}\right] d x
$$

Choosing to interpolate with cubic Legendre polynomials for all coordinates

$$
\begin{align*}
\bar{\alpha}_{i}(s) & =N(s)\left[\begin{array}{llll}
\bar{\alpha}_{p} & \bar{\alpha}_{p}^{\prime} & \bar{\alpha}_{q} & \bar{\alpha}_{q}^{\prime}
\end{array}\right]^{T}  \tag{29}\\
{\left[\begin{array}{c}
\overline{\boldsymbol{\alpha}} \\
\overline{\boldsymbol{\alpha}}^{\prime}
\end{array}\right] } & =\left[\begin{array}{c}
N(s) \\
B(s)
\end{array}\right] \boldsymbol{\alpha}_{p q}, \tag{30}
\end{align*}
$$

results in the internal energy and elemental stiffness matrix given as

$$
\begin{align*}
U & =\frac{1}{2} \boldsymbol{\alpha}_{p q}^{T} \int\left[\begin{array}{l}
N(s) \\
B(s)
\end{array}\right]^{T}\left[\begin{array}{cc}
\Phi & 0 \\
0 & \Phi
\end{array}\right]^{T}\left[\begin{array}{cc}
E & C \\
C^{T} & M
\end{array}\right]\left[\begin{array}{cc}
\Phi & 0 \\
0 & \Phi
\end{array}\right]\left[\begin{array}{c}
N(s) \\
B(s)
\end{array}\right] d s \boldsymbol{\alpha}_{p q}  \tag{31}\\
& =\boldsymbol{\alpha}_{p q}^{T} K_{\text {elem }} \boldsymbol{\alpha}_{p q} . \tag{32}
\end{align*}
$$

### 4.1 BOUNDARY CONDITIONS

For a fixed end, the boundary conditions become $\overline{\boldsymbol{u}}=\overline{\boldsymbol{u}}_{p}, \overline{\boldsymbol{\phi}}=\overline{\boldsymbol{\phi}}_{p}$, with the remaining coordinates zero. For a mixed boundary, we only have contact on a part of the cross section $\Omega_{c}$. Denote the separation at the boundary as $\Delta \boldsymbol{w}=\boldsymbol{w}^{L}-\boldsymbol{w}^{R}$. We may then define a penalty constraint by adding a penalty term related to $\int_{\Omega_{c}} \Delta \boldsymbol{w}^{T} \Delta \boldsymbol{w} d A$.

## 5 RESULTS

In this section we use the beam element outlined in section 4, with a truncated set of warping fields from VAM and EWM as obtained from section 2 and discussed in section 3, to model a notched leaf flexure and investigate number of required degrees of freedom for accurate results.
We consider a leaf flexure with a very small notch, as depicted in Figure 3. The ends of the leaf flexure are considered rigid, as they typically connect to a rigid structure. The right end is loaded with a force and 3 perpendicular torques, while the left end is considered fixed. In the middle of the leaf a small notch is present, which is a short section of beam with a reduced cross section. Sometimes such a notch is used in mechanism design to reduce the transverse bending stiffness of a leaf flexure, but not the shear stiffness. The notch presents a challenging case where highly local deformation is present.


Figure 3. The notched leaf flexure. The beam has a length of 4 m and the same cross section as discussed in section 2. At the notch, the width is reduced to $w=0.2 \mathrm{~m}$.

The notch is deliberately kept short to maximise the end effects. Furthermore the entire leaf flexure is meshed with 200 beam elements along its length, to ensure discretisation error is small. Beam variables are extracted from the finite element solution using Eq. (10).
Figure 4 shows the beam variables $u, \phi_{x}, \phi_{y}$ and $\phi_{z}$, as well as the traditional beam strains $\gamma, \kappa_{x}, \kappa_{y}, \kappa_{z}$ as a function of the span coordinate $s$. As can be seen, there are discontinuities in be beam variables occurring at the notch. The solutions based on a higher number of warping fields seem to be able to capture part of the discontinuity, although there is still a significant discrepancy visible. However, at the fixed boundaries, the warping element solutions converge well to the COMSOL solution for an increasing number of warping terms. Particularly, the torsional curvature $\kappa_{x}$ shows the predicted Vlasov decay from section 3 , with the traditional boundary condition $\kappa_{x}=0$. The bending curvature $\kappa_{y}$ also shows the decay predicted in section 3, allthough the effect is much less prominent as the decay mode is barely excited. Perhaps this explains why this effect is rarely mentioned in literature.
Note that even EWM $(\lambda=0)$ and VAM $(\mathscr{O}(1))$ solutions are able to produce decay solutions at the boundary, even though the decay solution is not explicitly included. This results from a restraint warping solution inherent to the beam model proposed in section 4 .
In order to give a more quantitative comparison, we compare the overall elongation, torsion and bending stiffness of the notched leaf flexure versus the number of included warping terms. The results are given in Figure 5. It can be seen that the beam element formulation from section 4 yields in general an overestimate of the stiffness. Increasing the number of warping fields leads to a better approximation of the true stiffness, although significant error remains when up to 38


Figure 4. The beam coordinates and beam strains over the length of the beam. The notch is located at $x=0$.
warping terms are included. No clear distinction is visible between the performance of VAM and the EWM.

## 6 CONCLUSIONS

In this work the use of a small set of warping coordinates in beam elements is investigated. Two expansions of warping fields have been considered. One based on the Variational Asymptotic Method (VAM), the other based on the Eigenwarping Method (EWM). VAM yields an expansion of the warping field that can be truncated based on the aspect ratio of the beam. The EWM yields an expansion of the warping field that can be truncated based on the decay length. Both methods show similarity in the obtained warping fields and are able to reproduce classical effects such as anticlasitic curvature and Vlasov restrained warping.
Only a few terms of both expansions are needed to describe the beam behaviour well in the continuum and at the rigid ends. However, in case of a severe discontinuity, such as the presented notched leaf flexure, a significant amount of additional terms are necessary to obtain accurate stiffness results. There is little difference between the two methods when looking at the number of terms required for a certain accuracy.

In this work we have introduced additional coordinates for each of the warping fields. Alternatives are possible by incorporating structural relations between the warping fields. However, this requires further research.


Figure 5. The elongation stiffness $k_{u}$, torsional stiffness $k_{\phi_{x}}$, and bending stiffnesses $k_{\phi_{y}}, k_{\phi_{z}}$ of the notched leaf flexure versus the used number of coordinates $N_{\alpha}$.

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