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Partitioning of Recursive Multibody Dynamics

Abhinandan Jain

Jet Propulsion Laboratory
California Institute of Technology
4800 Oak Grove Drive, Pasadena, California, USA
Abhinandan.Jain@jpl.nasa.gov

ABSTRACT

Minimal coordinate formulations provide the basis for the development of fast computational recursive methods for multibody dynamics. This paper studies the introduction of mass-less bodies to partition body degrees of freedom between the original and the new mass-less bodies. Such partitioning offers computational advantages by reducing the size of the matrix/vector quantities in the recursive algorithms. In particular, we study the relationship between the articulated body inertia forward dynamics algorithms for the unpartitioned and partitioned models of the system. We derive the relationship between the articulated body inertia quantities, and identify the agreements at the body level for the recursions. We furthermore continue a similar study for the operational space inertia quantities for the unpartitioned and partitioned formulations and derive relationships between them.

Keywords: recursive, algorithm, minimal coordinates, partitioned

1 INTRODUCTION

Recursive multibody methods are some of the most efficient computational methods available for multibody dynamics [1, 2, 3]. The normal Newton-Euler form of the recursive equations of motion of a serial chain multibody system has the form:

$$\begin{aligned} \underline{\mathcal{V}}(k) &= \underline{\phi}^*(k+1,k)\underline{\mathcal{V}}(k+1) + \underline{H}^*(k)\underline{\beta}(k) \\ \underline{\alpha}(k) &= \underline{\phi}^*(k+1,k)\underline{\alpha}(k+1) + \underline{H}^*(k)\underline{\dot{\beta}}(k) + \underline{\mathfrak{a}}(k) \\ \underline{\mathfrak{f}}(k) &= \underline{\phi}(k,k-1)\underline{\mathfrak{f}}(k-1) + \underline{M}(k)\underline{\alpha}(k) + \underline{\mathfrak{b}}(k) \end{aligned} \tag{1}$$

In the above, k denotes the body index, with body $k+1$ being inboard of body k as shown in Figure 1. $\underline{\mathcal{V}}(k)$ denotes the spatial velocity of the k^{th} body, $\underline{\alpha}(k)$ denotes the spatial velocity of the k^{th} body, $\underline{\mathfrak{f}}(k)$ denotes the spatial inter-body force between the k^{th} and $(k+1)^{th}$ bodies, $\underline{H}^*(k)$ joint map matrix for the k^{th} body, $\underline{\beta}(k)$ the generalized velocities for the k^{th} body, and body, $\underline{M}(k)$ the spatial inertia of the k^{th} body, $\underline{\mathfrak{a}}(k)$ the spatial Coriolis acceleration and $\underline{\mathfrak{b}}(k)$ the spatial gyroscopic force. Eq. 1 establishes the recursive structure by defining the relationship of key quantities between child and parent bodies. These form of the equations of motion is quite general and applies to systems with rigid bodies, flexible bodies, flexible joints and even closed-chain systems using constraint embedding [3]. In this paper, our notational convention will be to use underline for regular dynamics symbols such as $\underline{\phi}$ above to distinguish them from the *partitioned* quantities used below.

In this paper we will examine conditions under which we can introduce an intermediate fictitious k' body in between the physical k and $k+1$ bodies and allocate a subset of the $\underline{\beta}(k)$ generalized velocities for the k^{th} body to the new k' body and leave just the remaining ones with the original k^{th} body. That is, we partition the generalized velocities as

$$\underline{\beta}(k) = \begin{bmatrix} \beta(k'') \\ \beta(k') \end{bmatrix} \tag{2}$$

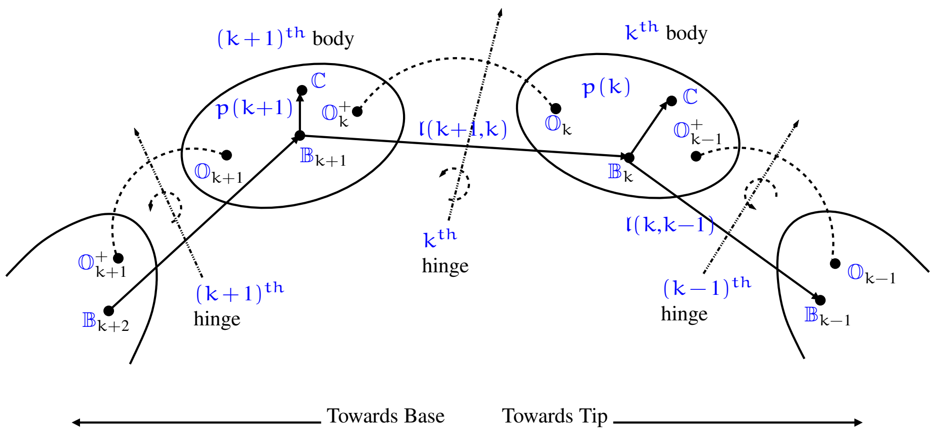

Figure 1. Illustration of bodies and the k^{th} hinge in a serial rigid body system

Introduction of this fictitious body also as the effect of partitioning the equations of motion as follows:

$$\begin{aligned}
\mathcal{V}(k') &= \phi^*(k+1,k')\mathcal{V}(k+1) + H^*(k')\beta(k') \\
\mathcal{V}(k) &= \phi^*(k',k)\mathcal{V}(k') + H^*(k'')\beta(k'') \\
\\
\alpha(k') &= \phi^*(k+1,k')\alpha(k+1) + H^*(k')\dot{\beta}(k') + \mathfrak{a}(k') \\
\alpha(k) &= \phi^*(k',k)\alpha(k') + H^*(k'')\dot{\beta}(k'') + \mathfrak{a}(k) \\
\\
\mathfrak{f}(k') &= \phi(k',k)\mathfrak{f}(k) + M(k')\alpha(k') + \mathfrak{b}(k') \\
\mathfrak{f}(k+1) &= \phi(k+1,k')\mathfrak{f}(k') + M(k+1)\alpha(k+1) + \mathfrak{b}(k+1)
\end{aligned} \tag{3}$$

While this partitioning increases the number of recursion steps, it also reduces the dimensionality of the individual steps. The reduction in size can be computationally beneficial when the size of the generalized velocities is large such as for flexible bodies.

In this paper, we examine such partitioned dynamics, and derive the conditions under which the k' pseudo-body can be added such that the regular equations of motion in Eq. 1 remain equivalent to the partitioned ones in Eq. 3, or more specifically where the partitioned values at the k^{th} body agree with the regular ones. Furthermore we derive the articulated body inertia recursive forward dynamics algorithm for the partitioned form of the equations of motion and study their relationship with the unpartitioned articulated body inertia quantities, and establish the areas of equivalence. We also carry out a similar analysis for the operational space inertias for the unpartitioned and partitioned cases. Though in this paper we choose to focus on serial-chain systems for the simplicity of exposition, the derivations easily apply to tree-topology multibody systems.

2 Partitioned Dynamics

We seek conditions under which the regular equations of motion in Eq. 1 are equivalent to the partitioned ones in Eq. 3, or more specifically where the partitioned values at the k^{th} body agree with the regular ones. The following Lemma establishes the conditions for equivalence.

Lemma 2.1. *The equivalence of the equations of motion represented by Eq. 1 and Eq. 3, defined by the expressions,*

$$\underline{\mathcal{V}}(k) = \mathcal{V}(k), \qquad \underline{\alpha}(k) = \alpha(k), \qquad \underline{\mathfrak{f}}(k) = \mathfrak{f}(k), \tag{4}$$

The equivalence holds if the following conditions are satisfied:

$$
\begin{aligned}
\underline{\phi}(k+1,k) &= \phi(k+1,k')\ \phi(k',k) \\
\underline{H}(k) &= \begin{bmatrix} H(k'') \\ H_C(k) \end{bmatrix} \quad \textit{where} \quad H_C(k) \triangleq H(k')\phi(k',k) \\
\underline{\mathfrak{a}}(k) &= \mathfrak{a}(k) + \phi^*(k',k)\mathfrak{a}(k') \\
\underline{M}(k) &= M(k), \qquad M(k') = 0 \quad \textit{and} \quad \mathfrak{b}(k') = 0
\end{aligned}
\tag{5}
$$

Proof: *Combining the spatial velocity equations in Eq. 3 leads to*

$$
\begin{aligned}
\mathcal{V}(k) &= \phi^*(k',k)\Big[\phi^*(k+1,k')\mathcal{V}(k+1) + H^*(k')\beta(k')\Big] + H^*(k'')\beta(k'') \\
&= \phi^*(k',k)\phi^*(k+1,k')\mathcal{V}(k+1) + \big[H^*(k''),\ \phi^*(k',k)H^*(k')\big]\begin{bmatrix}\beta(k'') \\ \beta(k')\end{bmatrix}
\end{aligned}
\tag{6}
$$

Comparing this with the spatial velocity expression in Eq. 1 we can see that they are equivalent if the conditions on $\underline{\phi}(k+1,k)$ *and* $\underline{H}(k)$ *in Eq. 5 are satisfied. A similar argument establishes that the spatial accelerations expressions additionally agree if the conditions on* $\underline{\mathfrak{a}}$ *in Eq. 5 agree.*

Similarly, the expressions for the spatial force agree if the conditions on $M(.)$ *in Eq. 5 hold. The* $\mathfrak{b}(k') = 0$ *requirement is a natural consequence of* $M(k') = 0$.

∎

The Eq. 5 equivalence conditions require that the fictitious k' body be *mass-less*. The partitioning process thus introduces a mass-less k' body between the $k+1$ and k bodies, and assigns some of the generalized velocities for the k^{th} body to the k' body and the remaining to the k^{th} body. We will refer to the set of conditions in Eq. 5 as the *Mass-less body partitioning assumption*. We now look at considerations effecting the partitioning choice.

- Firstly, when $\underline{\beta}(k) = 0$, we have $\underline{\mathcal{V}}(k) = \underline{\phi}^*(k+1,k)\underline{\mathcal{V}}(k+1)$ from Eq. 1 in the unpartitioned equations, and $\mathcal{V}(k') = \phi^*(k+1,k')\mathcal{V}(k+1)$ and $\mathcal{V}(k) = \phi^*(k',k)\mathcal{V}(k')$. This implies that in the absence of relative articulation velocities at the k^{th} hinge, the $\phi^*(*)$ matrices kinematically propagate the $\mathcal{V}(k+1)$ velocity from the $(k+1)^{th}$ body to the k' body, and finally on to the k^{th} body. This suggests that the k' zero-mass body be physically located between the $(k+1)^{th}$ and k^{th} bodies as shown in Figure 2.

  While technically we can use the identity matrix, or other choices, for either of the $\phi(k+1,k')$ and $\phi(k',k)$ that factorize $\underline{\phi}(k+1,k)$, the sparsity structure (or the lack) of $\underline{\phi}(k,k-1)$ usually determines the best choice for the factors. This is indeed the case for flexible bodies, where the location of the k' body separates the articulation degrees of freedom from the deformation degrees of freedom, and leading to nontrivial definitions of the $\phi(k+1,k')$ and $\phi(k',k)$ factors. However, when we apply partitioning across a sequence of articulation degrees of freedom (eg. for a universal or gimbal joint), there is usually no sparsity at this level, and the identity matrix is a reasonable choice for one of the factors.

- Once the location of the k' body has been made, we note that

  $$\phi^{-1}(k',k)\ \underline{H}^*(k) = \big[\phi^{-1}(k',k)H^*(k'),\ H^*(k'')\big]$$

  represents the unpartitioned joint map matrix for the k^{th} body transformed to the location of the new k' body. The partitioning of the generalized velocities thus comes down to apportioning columns of this matrix between the partitioned k and k' bodies. The partitioning is usually obvious from the physical nature and ordering of the generalized velocities and their

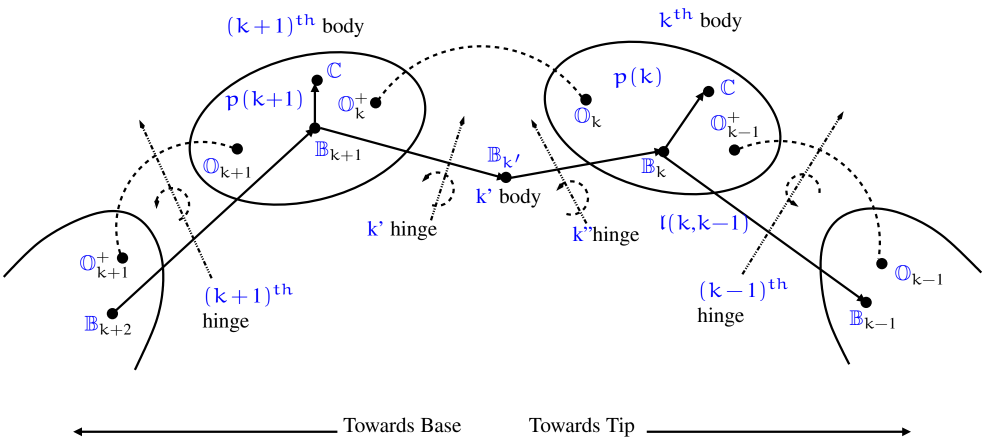

Figure 2. Illustration of partitioned system with k' body with k' and k'' hinges

associated generalized coordinates. When there is such ordering, the inboard generalized velocities should be assigned to the k' body and the outboard ones to the k^{th} body.

Additionally, the $\underline{H}^*(k)$ matrix at times has sparsity structure that suggests the best partitioning decision. For example, for flexible bodies, such sparsity leads to a natural partitioning between the articulation and modal deformation generalized velocities. While in principle, the partitioning process can continue to be applied and repeated, the advantages can become marginal without beneficial sparsity.

Examples of systems where such partitioning based decoupling is possible and useful are:

- Body hinges physically composed of a sequence of subhinges: Examples of these are *universal joints* and *gimbal* hinges, where the hinges are a sequence of single degree of freedom subhinges. The dynamics model can be formulated as one where there are massless bodies between these individual subhinges. One benefit of this decomposition is that while the $H^*(k)$ matrix for the original hinges varies as the coordinates change, the joint map matrix remains constant for the individual subhinges in the partitioned formulation. This case represents partitioning arising from the trivial partitioning of the $H(k)$ matrix and the generalized velocities.

- Multi-dof translation hinges - such as planar hinges - can also be decomposed into a sequence of single degree of freedom subhinges. This is also a fall back to the trivial case partitioning the $H(k)$ matrix and the generalized velocities.

- Flexible bodies have generalized velocities that are a combination of ones from the hinge and the remaining from the body deformation. We can decouple these equations of motion by introducing a a massless pseudo-body between the hinge and deformation degrees of freedom.

In this paper we will assume that the massless body partitioning assumption holds, and we will examine its impact on recursive dynamics algorithms for the system.

3 Partitioned ATBI dynamics

Assuming, that the partitioned dynamics equivalence conditions hold, in this section we explore the implications on the regular and partitioned forms of the articulated body inertia recursions [3].

These recursions are at the heart of the low-cost $O(\mathcal{N})$ forward dynamics algorithms for solving the equations of motion using minimal coordinate formulations for multibody systems. The general form of the articulated body inertia recursions for the unpartitioned system are:

$$
\begin{cases}
\qquad \underline{\mathcal{P}}^{+}(0) = \mathbf{0}, \quad \underline{\overline{\tau}}(0) = \mathbf{0} \\
\textbf{for k} \; = \; \mathbf{1} \cdots n \\
\quad \underline{\mathcal{P}}(k) = \underline{\phi}(k,k-1)\underline{\mathcal{P}}^{+}(k-1)\underline{\phi}^{*}(k,k-1) + \underline{M}(k) \\
\quad \underline{\mathcal{D}}(k) = \underline{H}(k)\underline{\mathcal{P}}(k)\underline{H}^{*}(k) \\
\quad \underline{\mathcal{G}}(k) = \underline{\mathcal{P}}(k)\underline{H}^{*}(k)\underline{\mathcal{D}}^{-1}(k) \\
\quad \underline{\tau}(k) = \underline{\mathcal{G}}(k)\underline{H}(k) \\
\quad \underline{\overline{\tau}}(k) = \mathbf{I} - \underline{\tau}(k) \\
\quad \underline{\mathcal{P}}^{+}(k) = \underline{\overline{\tau}}(k)\underline{\mathcal{P}}(k) \\
\textbf{end loop}
\end{cases}
\tag{7}
$$

On the other hand, for the partitioned dynamics model the articulated body inertia recursions are given by:

$$
\begin{cases}
\qquad \mathcal{P}^{+}(k'=0) = \mathbf{0} \\
\textbf{for k} \; = \; \mathbf{1} \cdots n \\
\quad \mathcal{P}(k) = \phi(k,k'-1)\mathcal{P}^{+}(k'-1)\phi^{*}(k,k'-1) + M(k) \\
\quad \mathcal{D}(k) = H(k'')\mathcal{P}(k)H^{*}(k'') \\
\quad \mathcal{G}(k) = \mathcal{P}(k)H^{*}(k'')\mathcal{D}^{-1}(k) \\
\quad \tau(k) = \mathcal{G}(k)H(k'') \\
\quad \overline{\tau}(k) = \mathbf{I} - \tau(k) \\
\quad \mathcal{P}^{+}(k) = \overline{\tau}(k)\mathcal{P}(k+1) \\
\\
\quad \mathcal{P}(k') = \phi(k',k)\mathcal{P}^{+}(k)\phi^{*}(k',k) \\
\quad \mathcal{D}(k') = H(k')\mathcal{P}(k')H^{*}(k') \\
\quad \mathcal{G}(k') = \mathcal{P}(k')H^{*}(k')\mathcal{D}^{-1}(k') \\
\quad \tau(k') = \mathcal{G}(k')H(k') \\
\quad \overline{\tau}(k') = \mathbf{I} - \tau(k') \\
\quad \mathcal{P}^{+}(k') = \overline{\tau}(k')\mathcal{P}(k') \\
\textbf{end loop}
\end{cases}
\tag{8}
$$

The following Lemma establishes the relationship between some of the unpartitioned and partitioned articulated body inertia quantities.

Lemma 3.1. *The expressions relating the regular articulated body inertia quantities to the partitioned articulated body inertia quantities are as below.*

1.

$$
\begin{aligned}
\underline{\mathcal{D}}(k) &= \begin{pmatrix} \mathcal{D}(k) & H(k'')\mathcal{P}(k)H_C^{*}(k) \\ H_C(k)\mathcal{P}(k)H^{*}(k'') & H_C(k)\mathcal{P}(k)H_C^{*}(k) \end{pmatrix} \\
\underline{\mathcal{D}}^{-1}(k) &= \begin{pmatrix} \mathcal{D}^{-1}(k) & 0 \\ 0 & 0 \end{pmatrix} + \\
&\qquad \begin{bmatrix} -\mathcal{G}^{*}(k)H_C^{*}(k) \\ \mathbf{I} \end{bmatrix} \mathcal{D}^{-1}(k') \left[-H_C(h)\mathcal{G}(k), \; \mathbf{I} \right]
\end{aligned}
\tag{9}
$$

2.

$$\underline{\mathcal{G}}(k) = \left[\mathcal{G}(k),\ \mathbf{0}\right] + \mathcal{P}^{+}(k)H_C^*(k)\mathcal{D}^{-1}(k')\left[-H_C(k)\mathcal{G}(k),\ \mathbf{I}\right] \tag{10}$$

3.

$$\begin{aligned}\underline{\tau}(k) &= \tau(k) + \mathcal{P}^{+}(k)H_C^*(k)\mathcal{D}^{-1}(k')H_C(k)\overline{\tau}(k)\\ \underline{\overline{\tau}}(k) &= \overline{\tau}(k) - \mathcal{P}^{+}(k)H_C^*(k)\mathcal{D}^{-1}(k')H_C(k)\overline{\tau}(k)\end{aligned} \tag{11}$$

Proof:

1. *The expression for* $\underline{\mathcal{D}}(k)$ *in Eq. 9 follows from*

$$\underline{\mathcal{D}}(k) \overset{7}{=} \underline{H}(k)\underline{\mathcal{P}}(k)\underline{H}^*(k) \overset{5}{=} \begin{bmatrix} H(k'') \\ H_C(k) \end{bmatrix} \underline{\mathcal{P}}(k) \left[H^*(k''),\ H_C^*(k)\right] \tag{12}$$

Now we use the following matrix inverse identity for a block-partitioned matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{bmatrix} -A^{-1}B \\ \mathbf{I} \end{bmatrix} F_2^{-1}\ \left[-CA^{-1},\ \mathbf{I}\right] \tag{13}$$

where $F_2 \overset{\triangle}{=} (D - CA^{-1}B)$ *(assuming that the* A *sub-matrix is invertible). Since* $\mathcal{D}^{-1}(k)$ *is invertible, we can apply this expression to the partitioned structure of* $\underline{\mathcal{D}}(k)$ *in Eq. 12 to get:*

$$\begin{aligned} F_2 &\overset{9}{=} H_C(k)\mathcal{P}(k)H_C^*(k) - H_C(k)\mathcal{P}(k)H(k'')\mathcal{D}^{-1}(k)H(k'')\mathcal{P}(k)H_C^*(k) \\ &\overset{8}{=} H_C(k)\mathcal{P}(k)H_C^*(k) - H_C(k)\mathcal{G}(k)H(k'')\mathcal{P}(k)H_C^*(k) \\ &\overset{8}{=} H_C(k)\mathcal{P}(k)H_C^*(k) - H_C(k)\tau(k)\mathcal{P}(k)H_C^*(k) \\ &\overset{8}{=} H_C(k)\overline{\tau}(k)\mathcal{P}(k)H_C^*(k) \overset{8}{=} H_C(k)\mathcal{P}^{+}(k)H_C^*(k) \\ &\overset{8}{=} H(k')\mathcal{P}k')H^*(k') \overset{8}{=} \mathcal{D}(k') \end{aligned}$$

and that $CA^{-1} = H_C(k)\mathcal{P}(k)H(k)\mathcal{D}^{-1}(k) = H_C(k)\mathcal{G}(k)$*. Using these and Eq. 13 with the partitioned expression for* $\underline{\mathcal{D}}(k)$ *in Eq. 9 leads to the expression there for* $\underline{\mathcal{D}}^{-1}(k)$.

2. *Now*

$$\begin{aligned} \underline{\mathcal{G}}(k) &\overset{7}{=} \underline{\mathcal{P}}(k)\underline{H}^*(k)\underline{\mathcal{D}}^{-1}(k) \\ &\overset{9,8}{=} \underline{\mathcal{P}}(k)\Big\{\left[H^*(k)\mathcal{D}^{-1}(k),\ \mathbf{0}\right] \\ &\qquad + \overline{\tau}^*(k)H_C^*(k)\mathcal{D}^{-1}(k')H_C(k)\left[-H_C(k)\mathcal{G}(k),\ \mathbf{I}\right]\Big\} \\ &\overset{8}{=} \left[\mathcal{G}(k),\ \mathbf{0}\right] + \mathcal{P}^{+}(k)H_C^*(k)\mathcal{D}^{-1}(k')\left[-H_C(k)\mathcal{G}(k),\ \mathbf{I}\right] \end{aligned}$$

3. *Note that*

$$\begin{aligned} \underline{\tau}(k) &\overset{7}{=} \underline{\mathcal{G}}(k)\underline{H}(k) \\ &\overset{10,8}{=} \tau(k) + \mathcal{P}^{+}(k)H_C^*(k)\mathcal{D}^{-1}(k')[-H_C(k)\tau(k) + H_C(k)] \\ &\overset{8}{=} \tau(k) + \mathcal{P}^{+}(k)H_C^*(k)\mathcal{D}^{-1}(k')H_C(k)\overline{\tau}(k) \end{aligned}$$

This establishes the first equation in Eq. 11. Additionally we have,

$$\underline{\overline{\tau}}(k) \overset{7}{=} \mathbf{I} - \underline{\tau}(k) \overset{11,8}{=} \overline{\tau}(k) - \mathcal{P}^{+}(k)H_C^*(k)\mathcal{D}^{-1}(k')H_C(k)\overline{\tau}(k)$$

establishing the second equation too.

■

The following Lemma derives useful identities that relate the unpartitioned and partitioned articulated body inertia quantities.

Lemma 3.2. *The following identities are true:*

1.
$$\phi(k',k)\ \underline{\mathcal{G}}(k) = \Big[\overline{\tau}(k')\phi(k',k)\mathcal{G}(k),\ \mathcal{G}(k')\Big] \tag{14}$$

2.
$$\begin{aligned}\phi(k',k)\ \underline{\tau}(k) &= \phi(k',k)\tau(k) + \tau(k')\phi(k',k)\overline{\tau}(k)\\ \phi(k',k)\ \underline{\overline{\tau}}(k) &= \overline{\tau}(k')\phi(k',k)\overline{\tau}(k)\end{aligned} \tag{15}$$

3.
$$\begin{aligned}\underline{H}^*(k)\underline{\mathcal{D}}^{-1}(k)\underline{H}(k) &= H^*(k'')\mathcal{D}^{-1}(k)H(k'')\\ &\quad + \overline{\tau}^*(k)H_C^*(k)\mathcal{D}^{-1}(k')H_C(k)\overline{\tau}(k)\end{aligned} \tag{16}$$

Proof:

1. *To establish Eq. 14 we have*

$$\begin{aligned}\phi(k',k)\underline{\mathcal{G}}(k) &\overset{10}{=} \Big[\phi(k',k)\mathcal{G}(k),\ \mathbf{0}\Big]\\ &\quad + \phi(k',k)\mathcal{P}^+(k)H_C^*(k)\mathcal{D}^{-1}(k')\Big[-H_C(k)\mathcal{G}(k),\ \mathbf{I}\Big]\\ &\overset{8}{=} \Big[\phi(k',k)\mathcal{G}(k),\ \mathbf{0}\Big]\\ &\quad + \mathcal{P}(k')H^*(k')\mathcal{D}^{-1}(k')\Big[-H_C(k)\mathcal{G}(k),\ \mathbf{I}\Big]\\ &\overset{8}{=} \Big[\phi(k',k)\mathcal{G}(k),\ \mathbf{0}\Big] + \mathcal{G}(k')\Big[-H_C(k)\mathcal{G}(k),\ \mathbf{I}\Big]\\ &\overset{8}{=} \Big[\phi(k',k)\mathcal{G}(k),\ \mathbf{0}\Big] + \Big[-\tau(k')\phi(k',k)\mathcal{G}(k),\ \mathcal{G}(k')\Big]\\ &\overset{8}{=} \Big[\overline{\tau}(k')\phi(k',k)\mathcal{G}(k),\ \mathcal{G}(k')\Big]\end{aligned}$$

2. *We have*

$$\begin{aligned}\phi(k',k)\ \underline{\tau}(k) &\overset{14,8}{=} \overline{\tau}(k')\phi(k',k)\tau(k) + \tau(k')\phi(k',k)\\ &= \phi(k',k)\tau(k) + \tau(k')\phi(k',k)\overline{\tau}(k)\end{aligned}$$

This establishes the first equation in Eq. 15. To establish the second one, just subtract both sides of this equation from $\phi(k',k)$ *and combine terms.*

3. *We have*

$$\begin{aligned}\underline{H}^*(k)\underline{\mathcal{D}}^{-1}(k)\underline{H}(k) &\overset{9,5}{=} H^*(k'')\mathcal{D}^{-1}(k)H(k'')\\ &\quad + \overline{\tau}^*(k)H_C^*(k)\mathcal{D}^{-1}(k')H_C(k) + \overline{\tau}(k)\end{aligned}$$

establishing Eq. 16.

■

The following Lemma shows that the articulated body inertia inertias computed by the unpartitioned and partitioned articulated body inertia algorithms agree with each other at specific points along the algorithm.

Lemma 3.3.

1. *The partitioned* $\underline{\mathcal{P}}(k)$ *and unpartitioned* $\mathcal{P}(k)$ *articulated body inertia quantities are the same for all bodies, i.e. for the* k^{th} *body we have*

$$\mathcal{P}(k) = \underline{\mathcal{P}}(k) \tag{17}$$

2. *Furthermore*

$$\underline{\mathcal{P}}^+(k) = \mathcal{P}^+(k) - \mathcal{P}^+(k)H_C^*(k)\mathcal{D}^{-1}(k')H_C(k)\mathcal{P}^+(k) \tag{18}$$

Proof:

1. *We will use a proof by induction approach. We know that* $M(1) = \mathcal{P}(1) = \underline{\mathcal{P}}(1)$*. To start the induction argument assume that* $\mathcal{P}(k) = \underline{\mathcal{P}}(k)$ *for some* k*. To complete the induction argument we need to show that* $\mathcal{P}(k+1) = \underline{\mathcal{P}}(k+1)$*. Now*

$$\begin{aligned}
\phi(k',k)\underline{\mathcal{P}}^+(k)\phi^*(k',k) &\stackrel{7}{=} \phi(k',k)\underline{\overline{\tau}}(k)\underline{\mathcal{P}}(k)\overline{\tau}^{**}(k)\phi^*(k',k) \\
&\stackrel{15}{=} \overline{\tau}(k')\phi(k',k)\overline{\tau}(k)\underline{\mathcal{P}}(k)\overline{\tau}^*(k)\phi^*(k',k)\overline{\tau}(k') \\
&= \overline{\tau}(k')\phi(k',k)\overline{\tau}(k)\mathcal{P}(k)\overline{\tau}^*(k)\phi^*(k',k)\overline{\tau}(k') \\
&\stackrel{8}{=} \overline{\tau}(k')\phi(k',k)\mathcal{P}^+(k)\phi^*(k',k)\overline{\tau}(k') \\
&\stackrel{8}{=} \overline{\tau}(k')\mathcal{P}(k')\overline{\tau}(k') \stackrel{8}{=} \mathcal{P}^+(k')
\end{aligned} \tag{19}$$

Now

$$\begin{aligned}
\underline{\mathcal{P}}(k+1) &\stackrel{7}{=} \underline{\phi}(k+1,k)\underline{\mathcal{P}}^+(k)\underline{\phi}^*(k+1,k) + \underline{M}(k+1) \\
&\stackrel{5}{=} \phi(k+1,k')\ \phi(k',k)\underline{\mathcal{P}}^+(k)\underline{\phi}^*(k',k)\phi^*(k+1,k') + \underline{M}(k+1) \\
&\stackrel{19}{=} \phi(k+1,k')\ \mathcal{P}^+(k')\phi^*(k+1,k') + \underline{M}(k+1) \stackrel{8}{=} \mathcal{P}(k+1))
\end{aligned}$$

This completes the proof by induction argument.

2. *We have*

$$\begin{aligned}
\underline{\mathcal{P}}^+(k) &\stackrel{7}{=} \underline{\overline{\tau}}(k)\underline{\mathcal{P}}(k) \stackrel{17}{=} \underline{\overline{\tau}}(k)\mathcal{P}(k) \\
&\stackrel{8}{=} \mathcal{P}^+(k) - \mathcal{P}^+(k)H_C^*(k)\mathcal{D}^{-1}(k')H_C(k)\mathcal{P}^+(k)
\end{aligned}$$

This establishes Eq. 18.

∎

This establishes that the articulated body inertia $\underline{\mathcal{P}}(k) = \mathcal{P}(k)$ for all bodies from the regular and partitioned approaches! Thus even though the other articulated body inertia recursion quantities across the regular and partitioned approaches are not the same, the articulated body inertia $\mathcal{P}(.)$ agree across both the regular and partitioned formulations or each body. This is in line with our expectation that the partitioning process should not effect the system dynamics itself.

4 Partitioned Operational Space Inertia

The operational space compliance matrix (OSCM) is an important dynamics quantity for a multi-body system [4, 5, 6, 7]. The OSCM Ω is defined as

$$\Omega \stackrel{\triangle}{=} \mathcal{J}\mathcal{M}^{-1}\mathcal{J}^* \tag{20}$$

where $\mathcal{M}$ denotes the mass matrix of the system, while $\mathcal{J}$ is the Jacobian to constraint/task space nodes of interest on the multibody system. The OSCM is required for handling bilateral and unilateral constraints on a multibody system as well for whole body motion control of robotic platforms. It has been shown that the OSCM can be decomposed as follows [7, 3]

$$\Omega = \Upsilon + \tilde{\psi}^* \Upsilon + \Upsilon \tilde{\psi} \tag{21}$$

where $\tilde{\psi}$ is lower block-triangular matrix associated with articulated body inertia recursions discussed earlier and Υ is a block-diagonal matrix. This structural decomposition result forms the basis for recursive methods for computing the OSCM matrix. In addition, the following base-to-tip recursion can be used to compute the block-diagonal terms of the Υ matrix [7, 3]:

$$\begin{aligned} \underline{\Upsilon}^+(k) &\stackrel{\triangle}{=} \underline{\phi}^*(k+1,k)\, \underline{\Upsilon}(k+1)\, \underline{\phi}(k+1,k) \\ \underline{\Upsilon}(k) &\stackrel{\triangle}{=} \underline{\overline{\tau}}^*(k)\, \underline{\Upsilon}^+(k)\, \overline{\tau}(k) + \underline{H}^*(k)\, \underline{\mathcal{D}}^{-1}(k)\, \underline{H}(k) \end{aligned} \tag{22}$$

The corresponding recursive algorithm for the block-diagonal OSCM terms for the partitioned model is as follows:

$$\begin{aligned} \Upsilon^+(k') &\stackrel{\triangle}{=} \phi^*(k+1,k')\, \Upsilon(k+1)\, \phi(k+1,k') \\ \Upsilon(k') &\stackrel{\triangle}{=} \overline{\tau}^*(k')\, \Upsilon^+(k')\, \overline{\tau}(k') + H^*(k')\, \mathcal{D}^{-1}(k')\, H(k') \\ \\ \Upsilon^+(k) &\stackrel{\triangle}{=} \phi^*(k',k)\, \Upsilon(k')\, \phi(k',k) \\ \Upsilon(k) &\stackrel{\triangle}{=} \overline{\tau}^*(k)\, \Upsilon^+(k)\, \overline{\tau}(k) + H^*(k'')\, \mathcal{D}^{-1}(k)\, H(k'') \end{aligned} \tag{23}$$

The following Lemma establishes the relationship between the $\underline{\Upsilon}(k)$ and $\Upsilon(k)$ unpartitioned and partitioned quantities.

Lemma 4.1.

1. *For any body* k*, the unpartitioned* $\underline{\Upsilon}(k)$ *and partitioned* $\Upsilon(k)$ *are the same, i.e.*

$$\underline{\Upsilon}(k) = \Upsilon(k) \tag{24}$$

2. *Also*

$$\underline{\Upsilon}^+(k) = \phi^*(k',k)\Upsilon^+(k')\phi(k',k) \tag{25}$$

Proof:

1. *We will use a proof by induction approach to establish Eq. 24. Assuming* $\underline{\Upsilon}(k+1) = \Upsilon(k+1)$*, we have*

$$\begin{aligned} \underline{\Upsilon}(k) &\stackrel{22}{=} \underline{\overline{\tau}}^*(k)\underline{\Upsilon}^+(k)\overline{\tau}(k) + \underline{H}^*(k)\underline{\mathcal{D}}^{-1}(k)\underline{H}(k) \\ &\stackrel{22}{=} \underline{\overline{\tau}}^*(k)\underline{\phi}^*(k+1,k)\underline{\Upsilon}(k+1)\underline{\phi}(k+1,k)\overline{\tau}(k)\underline{H}^*(k)\underline{\mathcal{D}}^{-1}(k)\underline{H}(k) \\ &\stackrel{5}{=} \underline{\overline{\tau}}^*(k)\phi^*(k',k)\phi^*(k+1,k')\Upsilon(k+1)\phi(k+1,k')\phi(k',k)\underline{\overline{\tau}}(k) \\ &\qquad + \underline{H}^*(k)\underline{\mathcal{D}}^{-1}(k)\underline{H}(k) \\ &\stackrel{15,16}{=} \overline{\tau}^*(k)\phi^*(k',k)\overline{\tau}^*(k')\Upsilon^+(k')k')\overline{\tau}(k')\phi(k',k)\overline{\tau}(k) \\ &\qquad + H^*(k'')\mathcal{D}^{-1}(k)H(k'') \\ &\qquad + \overline{\tau}^*(k)\phi^*(k',k)H^*(k')\mathcal{D}^{-1}(k')H(k')\phi(k',k)\overline{\tau}(k) \\ &\stackrel{23}{=} \overline{\tau}^*(k)\phi^*(k',k)\Upsilon(k')k')\phi(k',k)\overline{\tau}(k) + H^*(k'')\mathcal{D}^{-1}(k)H(k'') \\ &\stackrel{23}{=} \overline{\tau}^*(k)\Upsilon^+(k)\overline{\tau}(k) + H^*(k'')\mathcal{D}^{-1}(k)H(k'') \stackrel{23}{=} \Upsilon(k) \end{aligned}$$

2. *Note that*

$$\begin{aligned}\underline{\Upsilon}^{+}(k) &\stackrel{22}{=} \underline{\phi}^{*}(k+1,k)\underline{\Upsilon}(k+1)\underline{\phi}(k+1,k)\\ &\stackrel{5}{=} \phi^{*}(k',k)\phi^{*}(k+1,k')\Upsilon(k+1)\phi(k+1,k')\phi(k',k)\\ &\stackrel{23}{=} \phi^{*}(k',k)\Upsilon^{+}(k')\phi(k',k)\end{aligned}$$

5 CONCLUSIONS

This paper studies the class of tree-topology multibody systems satisfying the *mass-less partitioning assumption*. When this assumption holds, mass-less bodies can be introduced to partition the minimal coordinate degrees of freedom for the bodies in the system between the original and the new mass-less bodies. Such partioning offers computational advantages by reducing the size of the matrix/vector quantities involved in the recursive algorithms. In particular, we study the relationship between the articulated body inertia forward dynamics algorithms for the unpartitioned and partitioned models of the system. We derive the relationship between the articulated body inertia quantities, and identify agreements at the body level for the recursions. We furthermore continue a similar study for the OSCM quantities for the unpartitioned and partitioned formulations and derive relationships between them.

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