

# Bond-graph Representation of Ideal Constraints Via the Principle of Virtual Power

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## ABSTRACT

In this paper we develop bond-graph descriptions for ideal mechanical constraints, both embedded and adjoined, by stating and applying the principle of virtual power, at the particle level, to both types of constraints. The resulting system-level bond graph provides a bond-graphic derivation of Kane's equations for a general nonlinear nonholonomic system. **Keywords:** bond graph, virtual power, Kane's equations, ideal constraints, ideal machine.

## 1 Introduction

In a recent conference paper [1], the authors reported (but did not prove) that for a general system of constrained particles, having possible velocities represented by a partial velocity matrix, the partial velocity matrix is the modulus of a multibond graph [2] modulated transformer element, which represents the subsystem of embedded constraints described by the partial velocity matrix, in the sense that the effort outputs of the transformer element are the particle-level constraint forces for the embedded constraints. Although it is not the primary contribution of the paper, this fact is previously unreported in the literature.

In this paper we provide a proof of the above-described fact, using the principle of virtual power, which we state in a form well adapted to the system representation. We further extend the result by deriving, from the same principle, the bond-graph representation of an additional set of adjoined constraints, which could be in general rheonomic, nonholonomic, and nonlinear in velocity [3]. This result is also unreported in the literature.

By combining the so-derived subsystem bond graphs with an additional multibond subsystem, representing the particle kinetic energies, we develop a complete system-level bond graph (see Fig. 3), from which Kane's equations for the system are derived. Such a bond-graphic derivation of Kane's equations, for a general particle system incorporating both embedded constraints and adjoined nonlinear nonholonomic constraints, has not been previously reported in the literature.

## 2 System Description

We consider a complex<sup>1</sup> nonholonomic dynamical system in an inertial frame. In this frame, each system particle  $P$  has a position vector  $\mathbf{r}^P$ , determined by  $R$  generalized coordinates  $q_r$ ; let  $\mathbf{q}$  denote the column matrix of  $R$  generalized coordinates. Particle  $P$  also has a velocity vector  $\mathbf{v}^P$ , which is determined by  $\mathbf{q}$ , together with  $S \leq R$  generalized velocities<sup>2</sup>  $f_s$ ; let  $\mathbf{f}$  denote the column matrix of  $S$  generalized velocities. The number of embedded velocity constraints, which could be either holonomic<sup>3</sup> or linear nonholonomic, is

$$R_{ec} = R - S. \quad (1)$$

<sup>1</sup>I.e., subject to nonlinear velocity constraints.

<sup>2</sup>We use terminology from [4] where feasible; these are often called quasi-velocities in the literature.

<sup>3</sup>E.g. kinematic loop closure constraints.

For compatibility with bond-graph representation, we assume that all rheonomic constraints are adjoined explicitly, rather than embedded in the functional dependence of  $\mathbf{v}^P$  on  $\mathbf{q}$  and  $\mathbf{f}$ ; see Sec. 6.3. With this assumption,  $\mathbf{f}$  determines the velocity vectors  $\mathbf{v}^P$  according to

$$\mathbf{v}^P = \left( \frac{\partial \mathbf{v}^P}{\partial \mathbf{f}^\top} \right) \mathbf{f}, \quad (2)$$

and the generalized coordinate derivatives  $\dot{\mathbf{q}}$  according to

$$\dot{\mathbf{q}} = \mathbf{Q}\mathbf{f}, \quad (3)$$

where  $\mathbf{Q}$  is an  $R \times S$  matrix of rank  $S$ , and  $\partial \mathbf{v}^P / \partial \mathbf{f}$  is a column matrix of  $S$  nonholonomic partial velocity vectors, both of which depend only on  $\mathbf{q}$ . All constraints that are both scleronomic and either holonomic or simple (linear in velocity) nonholonomic are accounted for via constraint embedding with  $\mathbf{q}$  and  $\mathbf{f}$ .

There remain also  $L$  explicitly adjoined velocity constraints  $C_\ell$ , where  $C_\ell$  must be either rheonomic, nonlinear nonholonomic, or both. Without loss of generality, we assume there are  $N$  particles  $P_n$  in the system; with this assumption we may write  $C_\ell$  as<sup>4</sup>

$$C_\ell : \phi_\ell(\{\mathbf{v}_n\}, \mathbf{q}) = {}^P\phi_\ell(t), \quad (4)$$

where  $\mathbf{v}_n$  is shorthand notation for  $\mathbf{v}^{P_n}$ ,  $\{\mathbf{v}_n\}$  is a column matrix of  $N$  particle velocity vectors,  $\phi_\ell$  is a function of  $\{\mathbf{v}_n\}$  and  $\mathbf{q}$ , and  ${}^P\phi_\ell$  is a prescribed function of time  $t$  for adjoined rheonomic constraints, or zero for adjoined scleronomic constraints. On the assumption that the adjoined constraints are consistent and independent, the final constrained system has  $S - L$  degrees of freedom.

Each particle  $P_n$  is then subject to a total force resultant

$$\mathbf{R}_n = \mathbf{F}_n + {}^{ec}\mathbf{F}_n + \sum_\ell {}^{ac}\mathbf{F}_{n\ell}, \quad (5)$$

composed of the impressed force  $\mathbf{F}_n$ , the constraint force  ${}^{ec}\mathbf{F}_n$ , due to all embedded constraints, and the sum of the constraint forces  ${}^{ac}\mathbf{F}_{n\ell}$ , each due to the corresponding adjoined velocity constraint  $C_\ell$ .

### 3 Virtual Velocities and the Principle of Virtual Power

A virtual generalized velocity  $\delta_1 f_s$  is a hypothetical small (infinitesimal) change in the value of  $f_s$ , while holding  $t$  and  $\mathbf{q}$  constant [5]. Considering  $\delta_1 \mathbf{f}$  to be a length  $S$  column matrix of these changes, the resulting change in the value of  $\mathbf{v}_n$  is, from (2), the virtual particle velocity vector

$$\delta_1 \mathbf{v}_n = \left( \frac{\partial \mathbf{v}_n}{\partial \mathbf{f}^\top} \right) \delta_1 \mathbf{f}; \quad (6)$$

$\delta_1 \mathbf{v}_n$ , determined in this way from an arbitrary value of  $\delta_1 \mathbf{f}$ , will always satisfy all embedded constraints, by construction. Similarly, considering  $\{\delta_1 \mathbf{v}_n\}$  to be a column matrix of  $N$  virtual velocity vectors, the virtual constraint velocity  $\delta_1 \phi_\ell$  is defined as

$$\delta_1 \phi_\ell = \left( \frac{\partial \phi_\ell}{\partial \{\mathbf{v}_n\}^\top} \right) \cdot \{\delta_1 \mathbf{v}_n\}, \quad (7)$$

where  $\partial \phi_\ell / \partial \{\mathbf{v}_n\}$  is a column matrix of  $N$  basis-independent vector gradients  $\partial \phi_\ell / \partial \mathbf{v}_n$ . Because  ${}^P\phi_\ell(t)$  is independent of  $\mathbf{v}_n$ , Eq. (4) requires that virtual velocities  $\delta_1 \mathbf{v}_n$  consistent with the adjoined constraint  $C_\ell$  satisfy

$$\delta_1 \phi_\ell = \left( \frac{\partial \phi_\ell}{\partial \{\mathbf{v}_n\}^\top} \right) \cdot \{\delta_1 \mathbf{v}_n\} = 0. \quad (8)$$

A statement of the principle of virtual power which is well adapted to this classification of ideal constraints is as follows:

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<sup>4</sup>This representation is adapted from [3].

Considering the virtual particle velocities  $\delta_1 \mathbf{v}_n$  as defined above, the total virtual power of all virtual particle velocities consistent with all the embedded ideal constraints is zero, and the total virtual power of all virtual particle velocities consistent with each adjoined ideal constraint is zero.

Applying the principle first to the embedded constraints, the total virtual power of all embedded constraints is computed as

$$\delta_1 {}^{\text{ec}}P = \{\delta_1 \mathbf{v}_n\}^\top \cdot \{{}^{\text{ec}}\mathbf{F}_n\}; \quad (9)$$

and according to the principle as stated, this must be zero for all virtual velocities  $\delta_1 \mathbf{v}_n$  consistent with the embedded constraints. Using Eq. (6) for these virtual velocities, we find

$$0 = \delta_1 \mathbf{f}^\top \left( \frac{\partial \{\mathbf{v}_n\}^\top}{\partial \mathbf{f}} \right) \cdot \{{}^{\text{ec}}\mathbf{F}_n\}. \quad (10)$$

Since  $\delta_1 \mathbf{f}$  in (6) is arbitrary, we conclude

$$\mathbf{0} = \left( \frac{\partial \{\mathbf{v}_n\}^\top}{\partial \mathbf{f}} \right) \cdot \{{}^{\text{ec}}\mathbf{F}_n\}; \quad (11)$$

this equation leads to the standard unreduced<sup>5</sup> form of Kane's equations, when there are no adjoined constraints (i.e. (24), with  ${}^{\text{ac}}\mathbf{e} = \mathbf{0}$ ).

Now considering the adjoined constraints, the total virtual power of constraint  $C_\ell$ , summed over all particles  $P_n$ , is found as

$$\delta_1 {}^{\text{ac}}P_\ell = \{\delta_1 \mathbf{v}_n\}^\top \cdot \{{}^{\text{ac}}\mathbf{F}_{n\ell}\}, \quad (12)$$

and according to the principle of virtual power as stated above, this must be zero for all virtual velocities  $\delta_1 \mathbf{v}_n$  consistent with  $C_\ell$ . Making use of Eq. (8) for such virtual velocities, we find this will be identically zero if each constraint force  ${}^{\text{ac}}\mathbf{F}_{n\ell}$  is given by

$${}^{\text{ac}}\mathbf{F}_{n\ell} = \left( \frac{\partial \phi_\ell}{\partial \mathbf{v}_n} \right) \lambda_\ell, \quad (13)$$

where  $\lambda_\ell$  is a Lagrange multiplier required for dimensional consistency; i.e., the product of  $\phi_\ell$  and  $\lambda_\ell$  must have dimensions of power. The total constraint force for all adjoined constraints acting on  $P_n$  is then

$${}^{\text{ac}}\mathbf{F}_n = \sum_\ell {}^{\text{ac}}\mathbf{F}_{n\ell} = \left( \frac{\partial \boldsymbol{\phi}^\top}{\partial \mathbf{v}_n} \right) \boldsymbol{\lambda}, \quad (14)$$

where  $\boldsymbol{\lambda}$  is a column matrix of  $L$  Lagrange multipliers, and  $\boldsymbol{\phi}$  is a column matrix of  $L$  functions  $\phi_\ell$ .

#### 4 Constraint Power and Bond-Graph Interpretations

In this section we formulate the constraint power for both embedded and adjoined constraints, and we find that, by means of a bond-graph representation, they each can be represented as separate power-conserving subsystems, i.e. as ideal machines.

The total power of all embedded constraints is given by

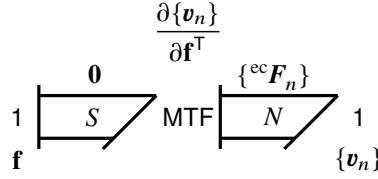
$${}^{\text{ec}}P = \{\mathbf{v}_n\}^\top \cdot \{{}^{\text{ec}}\mathbf{F}_n\}; \quad (15)$$

using (2) for  $\{\mathbf{v}_n\}$ , we see that this can also be written as

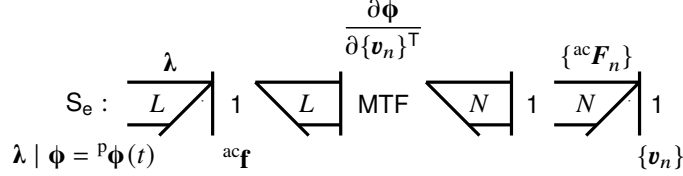
$${}^{\text{ec}}P = \mathbf{f}^\top \left( \frac{\partial \{\mathbf{v}_n\}^\top}{\partial \mathbf{f}} \right) \cdot \{{}^{\text{ec}}\mathbf{F}_n\},$$

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<sup>5</sup>By 'unreduced' is meant: not yet reduced to an explicit system of ordinary differential equations.



**Figure 1.** Bond-graph representation of embedded constraints



**Figure 2.** Bond-graph representation of adjoined constraints

or

$${}^{ec}P = \mathbf{f}^\top {}^{ec}\mathbf{e}, \quad (16)$$

where  ${}^{ec}\mathbf{e}$  is a column matrix of  $S$  generalized constraint forces given by

$${}^{ec}\mathbf{e} = \left( \frac{\partial \{v_n\}^\top}{\partial \mathbf{f}} \right) \cdot \{^{ec}F_n\}. \quad (17)$$

From (11) we also know that  ${}^{ec}\mathbf{e} = \mathbf{0}$ . Equations (2), (15)–(17) have the bond-graph representation illustrated in Fig. 1, where we see two 1-junctions (common flow, effort-summing), representing the flow matrices  $\mathbf{f}$  and  $\{v_n\}$  respectively, connected by a power-conserving multibond modulated TF element, which may be considered to be an ideal machine. The modulus of the TF element is given by the  $N \times S$  matrix of partial velocity vectors, i.e.  $\partial \{v_n\} / \partial \mathbf{f}^\top$ , which is a function of the generalized coordinates  $\mathbf{q}$ .

The total power of all adjoined constraints is given by

$${}^{ac}P = \{v_n\}^\top \cdot \{^{ac}F_n\}; \quad (18)$$

using (14) for  ${}^{ac}F_n$ , we see that this can also be written as

$${}^{ac}P = \{v_n\}^\top \cdot \left( \frac{\partial \Phi^\top}{\partial \{v_n\}} \right) \boldsymbol{\lambda},$$

or

$${}^{ac}P = {}^{ac}\mathbf{f}^\top \boldsymbol{\lambda}, \quad (19)$$

where  ${}^{ac}\mathbf{f}$  is a length  $L$  column matrix of generalized constraint velocities, given by

$${}^{ac}\mathbf{f} = \left( \frac{\partial \Phi}{\partial \{v_n\}^\top} \right) \cdot \{v_n\}. \quad (20)$$

Equations (14), (18)–(20) have the bond-graph representation illustrated in Fig. 2, where again we see a power-conserving multibond modulated TF element; but it is now actively driven by an effort source  $S_e$ , varying  $\boldsymbol{\lambda}$  with time to satisfy the adjoined constraints, and thereby supplying power to the system. This too may be considered to be an ideal machine. The modulus of the TF element is given by the  $L \times N$  matrix of function gradients  $\partial \Phi / \partial \{v_n\}^\top$ , which in the case of nonlinear nonholonomic constraints, may be a function of  $\mathbf{f}$ , as well as of  $\mathbf{q}$ . The TF element connects 1-junctions representing  $\{v_n\}$  and  ${}^{ac}\mathbf{f}$  respectively, and there is a third 1-junction, necessary to maintain the proper sign convention on the constraint efforts  $\{^{ac}F_n\}$ .

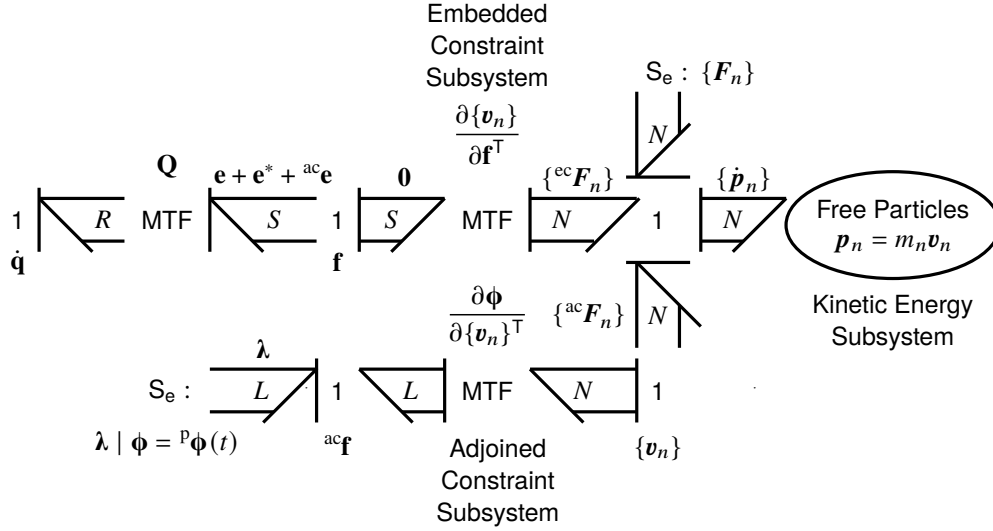


Figure 3. Bond-graph representation of full system

## 5 System-level Bond Graph and Equations of Motion

In this section we show how to assemble the constraint subsystem bond graphs developed above with the remaining components required to build a full system-level bond graph, leading to a derivation of Kane's equations for the system.

Given the inertial mass  $m_n$  for particle  $P_n$ , and the particle momentum vector defined as

$$\mathbf{p}_n = m_n \mathbf{v}_n, \quad (21)$$

Newton's second law provides a dynamic force balance on the particle, expressible as

$$\mathbf{F}_n + {}^{ec}\mathbf{F}_n + {}^{ac}\mathbf{F}_n = \dot{\mathbf{p}}_n. \quad (22)$$

This is converted to a system-level power balance, by dot-multiplying both sides of the equation by the particle velocity vector  $\mathbf{v}_n$ , and summing over all particles:

$$\{\mathbf{v}_n\}^T \cdot \{\mathbf{F}_n + {}^{ec}\mathbf{F}_n + {}^{ac}\mathbf{F}_n\} = \{\mathbf{v}_n\}^T \cdot \{\dot{\mathbf{p}}_n\}; \quad (23)$$

here we see the total rate at which work is being performed on the system on the left, and the total rate at which kinetic energy is increasing on the right. Equations (21)–(23) are represented by the kinetic energy subsystem on the right side of Fig. 3, which is combined with the constraint subsystems previously introduced in Figs. 1–2, as well as an MTF element with matrix modulus  $\mathbf{Q}$ , representing the velocity transformation from  $\mathbf{f}$  to  $\dot{\mathbf{q}}$  (Eq. (3)), to form a complete system-level multibond graph.

Fig. 3 allows a bond-graphic derivation of Kane's equations for the system, in unreduced form. Note that the kinetic energy subsystem is in full differential causality [6], with its inputs being the individual particle velocities  $\mathbf{v}_n$ , while its outputs are the individual particle inertial forces  $-\dot{\mathbf{p}}_n$ . The equations are found by propagating the system effort inputs  $\{\mathbf{F}_n\}$ ,  $\boldsymbol{\lambda}$ , and  $\{\dot{\mathbf{p}}_n\}$  separately through the system, using the effort transformations indicated by the constitutive equations of the modulated TF elements, to the left side of the  $\mathbf{f}$  1-junction, and summing the resulting generalized force terms there. The effort balance at the 1-junction is then

$$\mathbf{e} + \mathbf{e}^* + {}^{ac}\mathbf{e} = \mathbf{0}, \quad (24)$$

where

$$\mathbf{e} = \left( \frac{\partial \{\mathbf{v}_n\}^T}{\partial \mathbf{f}} \right) \cdot \{\mathbf{F}_n\} \quad (25)$$

represents a column matrix of generalized impressed forces,

$$\mathbf{e}^* = \left( \frac{\partial \{\mathbf{v}_n\}^T}{\partial \mathbf{f}} \right) \cdot \{-\dot{\mathbf{p}}_n\} \quad (26)$$

represents a column matrix of generalized inertia forces, and

$${}^{\text{ac}}\mathbf{e} = \left( \frac{\partial \{\mathbf{v}_n\}^T}{\partial \mathbf{f}} \right) \cdot \left( \frac{\partial \{\phi_\ell\}^T}{\partial \{\mathbf{v}_n\}} \right) \boldsymbol{\lambda} \quad (27)$$

represents a column matrix of generalized adjoined constraint forces, all of length  $S$ . This balance of generalized forces can be straightforwardly reduced to a set of  $S$  ordinary differential equations for  $\mathbf{f}$ , along with a set of  $L$  algebraic equations for  $\boldsymbol{\lambda}$ ; the details however are beyond the intended scope of this paper.

## 6 Discussion

### 6.1 Generalizations

The development above can be generalized to a system composed of point masses, rigid bodies, and flexible bodies, by considering each rigid or flexible body to consist of a large but finite number of particles. For a rigid body, the particle impressed forces will sum to a total impressed force through a reference point [3, Sec 2.4], and a total moment about the reference point, the reference point being fixed in the body frame, usually at the center of mass. In such a development, the vector velocities for all points in a rigid body, in any adjoined velocity constraint involving that body, will be replaced by two velocity vectors per rigid body: the angular velocity vector of the body, and the linear velocity vector of the associated reference point, both in the inertial reference frame. An adjoined velocity constraint involving a flexible body may need to incorporate additional generalized velocities representing elastic degrees of freedom associated with the body.

### 6.2 Nonlinear nonholonomic constraints

A nonlinear nonholonomic constraint generally reflects an actively imposed system control objective; it cannot be created or imposed by purely mechanical means [7, 8]. The energy cost of imposing such a constraint is then of great interest. This cost is found by integrating the product of the associated Lagrange multiplier and the associated generalized constraint velocity, over a time period of interest.

Any nonlinear nonholonomic constraint could be embedded in the system description at the acceleration level, rather than adjoined at the velocity level [3]. This would eliminate the associated generalized force (Lagrange multiplier) from the formulation. As we have seen however, the Lagrange multiplier is actually necessary in order to compute the power required by the system to enforce the constraint; nonlinear nonholonomic constraints are therefore adjoined to, rather than embedded in, the initial system description.

### 6.3 Rheonomic constraints

Conventional developments of analytical dynamics allow rheonomic constraints to be embedded in the system description, thereby eliminating explicitly time-dependent generalized coordinates from the system description, and specifying that the particle positions are direct functions of time as well as of the set of remaining independent generalized coordinates. See e.g. [9, Sec. 2.9]. While logically self-consistent, such an approach is fundamentally incompatible with the bond-graph methodology, because it makes it impossible to identify on a bond graph the physical origins of power delivered to the system by the rheonomic constraints, which is changes in the time-dependent generalized coordinates.

Our approach here, which is not less general, is to adjoin all rheonomic constraints via Lagrange multipliers, after the system has been completely described in terms of a full set of generalized

coordinates, including those that will become time-dependent when the rheonomic constraints are enforced. This allows the physical source of power delivered by a rheonomic constraint to be identified, enabling a calculation of the energy cost of the constraint over a given time interval. An alternative to adjoining rheonomic constraints with Lagrange multipliers is to employ constraint-adapted generalized velocities in the  $\mathbf{f}$  matrix, together with the method of constraint relaxation [1, 10]. The constraint reactions, formerly supplied by the Lagrange multipliers, then become available at the end of the analysis, when the constraints are enforced.

#### 6.4 Advantages of bond graph representations

Bond graphs, being based on Heaviside's principle of the continuity of power flow in space<sup>6</sup> [11], provide a power-port based network description of multiphysics systems, using a small number of ideal elements, thus revealing the topology of system power flows. This paper uses the multibond-graph dialect [2], with an extension to allow basis-free vectors as power-conjugate efforts and flows on multibonds [12]. By representing a constrained mechanical system at a high level of abstraction, the bond graph provides a system description that aggregates subsystems by phenomenology, rather than strictly by physical location, providing a high-level overview of system power flows.

A unique contribution of the bond-graph methodology in this paper is the representation of the particle-level constraint efforts and flows, for all embedded constraints and each adjoined constraint, as modulated multibond transformer elements, i.e. as ideal machines. The correspondence between ideal constraints and ideal machines has been recognized from antiquity, e.g. as in Archimedes's 'law of the lever', but the bond-graph representation used here makes this correspondence both formal and rigorous, via the principle of virtual power.

An interesting conclusion that may be drawn from the resulting system-level bond graph in Fig. 3 is that the particle-level constraint forces  ${}^{ec}\mathbf{F}_n$  and  ${}^{ac}\mathbf{F}_n$  are indeed *internal* forces from the perspective of the topology of system power flow; this parallels the often noted but less general observation that the (geometrically) internal forces in a rigid body are forces of constraint. A related observation is that Newton's third law never needs to be invoked in order to create a bond-graph model of a mechanical system; the more general principle of continuity of power flow being sufficient to replace it.

#### 6.5 Advantages of the principle of virtual power

The variational principles of mechanics can generally be categorized as zeroth order (virtual work, least action), first order (virtual power), or second order (least constraint). Since the zeroth order principles are not powerful enough to provide a solution for systems with nonlinear nonholonomic constraints [13], it then develops that the principle of virtual power is the variational principle of minimum order that provides a solution for systems with nonlinear nonholonomic constraints, while also providing solutions for systems with constraints that could be solved via a zeroth order principle. It can be applied without difficulty to problems of statics, replacing the principle of virtual work for this purpose. Indeed, its use in the field of continuum mechanics has been steadily growing, supporting e.g. the development of theories of microstructured media [14]. As this paper demonstrates, it is also highly compatible with mechanical system bond-graph representations.

#### 6.6 Kane's equations

Although originally developed without reference to any variational principle [15, 16, 9], it has been recognized for some time that Kane's equations are readily derived from the principle of virtual power [17, 5]. This paper strengthens that conclusion by providing a particle-level bond-graphic derivation of Kane's equations for a complex nonholonomic system, the underlying bond graph having been developed by application of the virtual power principle separately to embedded and adjoined constraint subsystems.

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<sup>6</sup>"When energy goes from place to place it traverses the intermediate space."

Karnopp in 1992 concluded that the pairing of a high-level bond-graph representation with a powerful equation-based methodology, such as Lagrange's or Hamilton's equations, provides an effective solution to the problem of differential causality in bond-graph models of constrained mechanical systems [6]. These methodologies are, however, limited to holonomic systems. Kane's equations, as illustrated above, provide a more comprehensive alternative methodology, which can handle embedded nonholonomic constraints, via the use of generalized velocities (quasi-velocities) as flow variables. The bond-graphic derivation of Kane's equations shown above proceeds by representing *all* system particles with differential causality, which solves Karnopp's problem of differential causality by embracing it.

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